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Singularities of Fourier transforms: A tribute to M. J. Lighthill

JAMES LIGHTHILL 2000 MEMORIAL PAPER

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Abstract. Some singular generalised functions of several variables are introduced and their properties described. They are used to support some general theorems to the effect that (a) the singularities of a generalised function are responsible for the behaviour of its Fourier transform at infinity and (b) the behaviour of a generalised function at infinity determines the singularities of its Fourier transform.

Key words: generalised functions, distributions, integral transforms, asymptotics.

1. Introduction

When I was on the staff of the University of Manchester the Department of Applied Mathematics held seminars in the afternoon of every Monday and Friday (and Saturday sometimes). Attendance at these seminars by staff and post-graduate students was strongly advised *i.e.* compulsory. The intention was to make members of the department aware of the latest advances throughout applied mathematics (e-mail had yet to be invented) and many international figures presented the lectures. It was a marvellous experience for an Assistant Lecturer except on those terrifying days when one was the designated speaker.

As a result of these seminars we knew that rigorous theories of the δ-function were being developed by Schwartz, Mikusiński, Silva and others. But none of these theories seemed suitable for the applied field (this was well before weak solutions became all the rage). Then one day George Temple gave a seminar which explained his ideas for an appropriate starting point for applied mathematicians. James Lighhill saw their importance immediately and set about constructing a working tool. Shortly thereafter his book *An Introduction to Fourier Analysis and Generalised Functions* [1] was published. I do not know how long it took James to write the book but I suspect that it was only a few days. At that time the speed and depth of his thinking appeared supersonic to those of us who proceeded at a subsonic pace. (He was a redoubtable player in the games at the annual Christmas party once he had finished his stint at the piano.) If you floated an idea before him on Friday and it took his fancy you could easily find that he had worked up a fully developed theory by the following Monday. The short time which elapsed between the Temple seminar and the appearance of the book precludes any long preparation. Yet the contents of the book have proved so valuable that it is still in print over 40 years later.

The book by James deals with generalised functions of a single variable and I decided to try to extend his ideas to several variables. The result was *Generalised Functions* [2] published in 1966. For both books the definitions of certain generalised functions had to be settled. Singular generalised functions are of great importance in applications. They occur in the force-fields of point sources and in the propagation of wavefronts. They may arise also in the solution of the differential form of boundary value problems. However, their definition is not free from ambiguity and a symbol may have different meanings for different writers. When singular generalised functions are involved in formulae it is always necessary to check the defintions being employed.

The reason why different definitions occur can be explained as follows. Suppose that, in one dimension, the generalised function x_+^{-1} is to be defined. It is equal to $1/x$ for $x > 0$ and 0 for $x < 0$. Since any generalised function can be multiplied by x it would be expected that

$$x.x_+^{-1} = H(x) \tag{*}$$

where $H(x)$ is the Heaviside step friction which is 1 for $x > 0$ and 0 for $x < 0$. It is permissible to take a derivative by the usual product rule and so

$$x_+^{-1} + x.(x_+^{-1})' = \delta(x) \tag{**}$$

It would be convenient if x_+^{-1} obeyed rules similar to those for $1/x$ so that $(x_+^{-1})' = -x_+^{-2}$ and $x.x_+^{-2} = x_+^{-1}$. Unfortunately, that leads to an inconsistency in (**) because the left-hand side vanishes then.

A similar difficulty is present in three dimensions. The quantity $(x^2 + y^2 + z^2)^{-3/2}$ and its derivatives turn up in electricity and magnetism when dealing with point sources and multipoles. If $(x^2 + y^2 + z^2)^{-1/2}$ is written as r the analogue of (*) is

$$r^2.r^{-3} = r^{-1}.$$

Application of the Laplacian ∇^2 to this relation leads to

$$6r^{-3} + 4\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)r^{-3} + r^2 \cdot \nabla^2 r^{-3} = -4\pi\delta(x)\delta(y)\delta(z).$$

If now it is asked that the usual formulae $\nabla^2 r^{-3} = 6r^{-5}$, $r^2.r^{-5} = r^{-3}$ and

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)r^{-3} = 3r^{-3}$$

hold there is an inconsistency again.

It may be mentioned that in four dimensions, a similar problem arises for $(c^2t^2 - x^2 - y^2 - z^2)^{-2}$ but details will be omitted.

Thus, it appears that, for a single variable, there is no way of defining x_+^{-m}, with m a positive integer, so that both the standard rule for the derivative $(x_+^{-m})' = -mx_+^{-m-1}$ and the standard rule for multiplication $x.x_+^{-m-1} = x_+^{-m}$ are retained. Moving to higher dimensions does not resolve matters.

The method which James devised to surmount this hurdle was to regard $x_+^{-1} + C\delta(x)$, for any constant C, as x_+^{-1} also; then (*) remains valid since $x.\delta(x) = 0$. Furthermore, $x_+^{-2} = (x_+^{-1})' + C_1\delta'(x)$ for any constant C_1 in the Lighthill theory. The presence of the arbitrary constants renders (**) consistent with $x.x_+^{-2} = x_+^{-1}$

I adopted the same device in my book. However, it became clear from the reaction of readers that they found this a confusing procedure especially when handling more complicated singular generalised functions. So, when I came to write a new version [3] of my book, I decided to abandon the Lighthill pattern and choose between multiplication and the derivative.

I opted to retain the usual rule for multiplication so that $x.x_+^{-m} = x_+^{1-m}$. The price was a more complex rule for the derivative since it was no longer possible to have $(x_+^{-m})' = -mx_+^{-1-m}$. Of course, it is perfectly possible to have a consistent theory which retains the rule for the derivative at the expense of multiplication. The same notation may mean different things for different authors. When a singular generalised function is encountered it is always essential to check what definition the author is employing.

Not all singular generalised functions require special attention. Some, such as x^{-m}, are in common currency and comply with the standard rules for both multiplication and the derivative. On the other hand, $|x|^{-m}$ does not. It will be seen subsequently that basically one technique is deployed in defining generalised functions which can exhibit singularities. Suppose that a generalised function depends on the parameter β, becoming singular for some range of β. In essence, definitions are extended from values of β where the generalised function is conventional by analytic continuation in the complex β-plane. If the analytic continuation covers all values of β no special treatment is necessary. Only where the analytic continuation fails have additional definitions to be supplied.

The importance of singular generalised functions and their Fourier transforms stems from their occurrence in a variety of applications. Some applications have been alluded to already. Others include the Hadamard finite part, Liouville fractional derivatives, splines and the theory of wavelets to mention a few examples.

In the following pages, where I have attempted to extend the theory of James on the singularities of Fourier transforms of a single variable to several variables, I have retained multiplication in standard form. Apart from this deviation from his way of doing things, I hope that my treatment is in line with what James would have done, though probably his exposition would have been more elegant and illuminating.

Section 2 contains a brief reminder of the Temple-Lighthill theory of generalised function together with a discussion of properties of the powers r^β of the radial distance. In Section 3 the generalised function $r^\beta \log^m r$ is handled. The Fourier transforms of r^β and $r^\beta \log^m r$ are obtained in Section 4.

More complicated singularities can occur in generalised functions of several variables than those of a single variable. They need not be confined to a single point. An example is given in Section 5 where the generalised functions are singular on a boundary.

The preceding theory is adapted in Section 6 to the Bessel generalised function $r^\nu J_\nu(r)$. Its properties are studied and its Fourier transform derived.

Section 7 is devoted to an investigation of the relation between a generalised function and its Fourier transform. On the one hand it is shown that, under fairly general conditions, the singularities of a generalised function dictate the behaviour of its transform at infinity. On the other hand the behaviour of the generalised function at infinity controls the singularities of its transform. The theory is illustrated by means of examples from the earlier sections.

An appendix contains some material which is needed in Section 7. It has been separated off to avoid an undue interruption of the argument in Section 7.

2. Singular generalised functions in R_n

A typical point of R_n will be identified by the vector $\mathbf{x}$ or by the Cartesian components x_1, $x_2, \dots, x_n$. The radial distance will be denoted by r so that $r = (x_1^2 + x_2^2 + \dots x_n^2)^{1/2}$. The generalised partial derivative $\partial/\partial x_i$ will be abbreviated to ∂_i; it may be taken as the conventional partial derivative in appropriate circumstances. The Laplacian ∇^2 wil signify the usual operator generalised *i.e.*

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \ldots + \partial_n^2.$$

An integral over the whole of R_n will be written

$$\int_{-\infty}^{\infty} g(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The basis of the Temple-Lighthill theory of generalised functions is the notion of a good function as Lighthill called it. A good function $\gamma(\mathbf{x})$ is infinitely differentiable everywhere on R_n and

$$\lim_{r\to\infty} \left| r^k \partial_1^{p_1} \partial_2^{p_2} \ldots \partial_n^{p_n} \gamma(\mathbf{x}) \right| = 0$$

for every radial direction, every integer $k \geq 0$ and all integers $p_1 \geq 0, p_2 \geq 0, \ldots, p_n \geq 0$. A generalised function $g(\mathbf{x})$ is defined by a sequence $\{\gamma_m(\mathbf{x})\}$ of good functions such that

$$\int_{-\infty}^{\infty} g(\mathbf{x})\gamma(\mathbf{x})\,\mathrm{d}\mathbf{x} = \lim_{m\to\infty} \int_{-\infty}^{\infty} \gamma_m(\mathbf{x})\gamma(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

for every good γ, provided that the limit exists. All sequences which give the same limit are regarded as equivalent and define the same generalised function.

Since the derivative of a good function is good the sequence $\{\partial_j \gamma_m(\mathbf{x})\}$ defines the generalised derivative $\partial_j g(\mathbf{x})$. The property

$$\int_{-\infty}^{\infty} \{\partial_j g(\mathbf{x})\}\gamma(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_{-\infty}^{\infty} g(\mathbf{x})\partial_j \gamma(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

follows from

$$\int_{-\infty}^{\infty} \{\partial_j \gamma_m(\mathbf{x})\}\gamma(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_{-\infty}^{\infty} \gamma_m(\mathbf{x})\partial_j \gamma(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

It can be shown that any generalised function can be expressed in terms of generalised derivatives via

$$g(\mathbf{x}) = \partial_1^{p_1} \partial_2^{p_2} \ldots \partial_n^{p_n} f(\mathbf{x}),$$

where the conventional $f(\mathbf{x})$ is continuous on R_n and

$$\int_{-\infty}^{\infty} \{|f(\mathbf{x})| / (1 + r^2)^N\}\,\mathrm{d}\mathbf{x} < \infty$$

for some finite N. This offers an alternative route for defining a generalised function.

Multiplication is less straightforward because the product of two generalised functions is undefined except in special circumstances. For the purpose of multiplication fairly good functions are introduced. The function $\beta(\mathbf{x})$ is fairly good when it is infinitely differentiable on R_n and, together with all its derivatives, is bounded at infinity by r^N for some finite N. Since $\beta(\mathbf{x})\gamma(\mathbf{x})$ is good the sequence $\{\beta(\mathbf{x})\gamma_m(\mathbf{x})\}$ defines $\beta(\mathbf{x})g(\mathbf{x})$ with the property

$$\int_{-\infty}^{\infty} \{\beta(\mathbf{x})g(\mathbf{x})\}\gamma(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} g(\mathbf{x})\{\beta(\mathbf{x})\gamma(\mathbf{x})\}\,\mathrm{d}\mathbf{x}.$$

In other words it is possible to multiply any generalised function by a fairly good function.

Note that x_j and r^2 are fairly good but r is not. Hence the multiplication undertaken in the introduction is justified.

Moreover

$$\int_{-\infty}^{\infty} \partial_j(\beta g)\gamma \, d\mathbf{x} = -\int_{-\infty}^{\infty} \beta g \partial_j \gamma \, d\mathbf{x} = -\int_{-\infty}^{\infty} g\{\partial_j(\beta\gamma) - \gamma\partial_j\beta\} \, d\mathbf{x}$$
$$= \int_{-\infty}^{\infty} (\beta\partial_j g + g\partial_j\beta)\gamma \, d\mathbf{x},$$

verifying the standard rule for the generalised derivative of a product of a fairly good function and a generalised function. This confirms the validity of the derivatives calculated in the introduction.

A sequence $\{g_\mu(\mathbf{x})\}$ of generalised functions possesses a generalised limit $g(\mathbf{x})$ as $\mu \to 0$ if, and only if,

$$\lim_{\mu\to 0} \int_{-\infty}^{\infty} g_\mu(\mathbf{x})\gamma(\mathbf{x}) \, d\mathbf{x} = \int_{-\infty}^{\infty} g(\mathbf{x})\gamma(\mathbf{x}) \, d\mathbf{x}$$

for every good $\gamma(\mathbf{x})$. The generalised limit will be denoted by $\operatorname{Lim}_{\mu\to 0} g_\mu(\mathbf{x})$ in order to avoid any confusion with the conventional limit $\lim_{\mu\to 0} g_\mu(\mathbf{x})$ (which may or may not exist when the generalised limit does).

It is transparent that the generalised limit coincides with the conventional limit when the conventional one exists. In addition, when the generalised limit exists it is an easy deduction that

$$\operatorname*{Lim}_{\mu\to 0} \partial_j g_\mu(\mathbf{x}) = \partial_j g(\mathbf{x}),$$

i.e. a derivative and the generalised limit can be interchanged. It is also a simple matter to include a linear change of variable so that

$$\operatorname*{Lim}_{\mu\to 0} g_\mu(a_1x_1 + b_1, \ldots, a_nx_n + b_n) = g(a_1x_1 + b_1, \ldots, a_nx_n + b_n)$$

with $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ constants. Another result is

$$\operatorname*{Lim}_{\mu\to 0} \beta(\mathbf{x})g_\mu(\mathbf{x}) = \beta(\mathbf{x})g(\mathbf{x}),$$

so that it is always permissible to interchange the generalised limit and multipliation by a fairly good function.

Points in the Fourier transform space will be denoted by the vector $\boldsymbol{\alpha}$ with components $\alpha_1, \alpha_2, \ldots, \alpha_n$. The letter α will be used for the radial distance so that $\alpha = (\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2)^{1/2}$. A Fourier transform will be indicated usually by a capital letter *e.g.*

$$\Gamma(\boldsymbol{\alpha}) = \int_{-\infty}^{\infty} e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\gamma(\mathbf{x}) \, d\mathbf{x}.$$

Since $\Gamma(\boldsymbol{\alpha})$ is good $\{\Gamma_m(\boldsymbol{\alpha})\}$ defines a generalised fimction $G(\boldsymbol{\alpha})$, the Fourier transform of $g(\mathbf{x})$, and

$$G(\boldsymbol{\alpha}) = \int_{-\infty}^{\infty} e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}} g(\mathbf{x}) \, d\mathbf{x}.$$

The Fourier inversion theorem then takes the form

$$g(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{i\boldsymbol{\alpha}\cdot\mathbf{x}} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}.$$

It follows at once that $\mathrm{Lim}_{\mu\to 0}\, g_\mu(\mathbf{x}) = g(\mathbf{x})$ implies

$$\mathop{\mathrm{Lim}}_{\mu\to 0} G_\mu(\boldsymbol{\alpha}) = G(\boldsymbol{\alpha}).$$

Evidently, the generalised limit is a much more powerful device than the conventional limit. Not only can it exist when the conventional one does not but also it enjoys a much greater flexibility. Its properties, as described above, will be used freely in the following pages without special mention.

An infinite series $\sum_m^\infty g_m(\mathbf{x})$ is said to be generally convergent when $\mathrm{Lim}_{M\to\infty} \sum_m^M g_m(\mathbf{x})$ exists. Clearly, a corvergent series is generally convergent. Also, from the above properties of the generalised limit, derivatives and Fourier transforms can be taken term-by-term provided suitable precautions are observed (see Jones [3, Chapter 5]).

The conventional function r^β satisfies

$$\nabla^2 r^\beta = \beta(\beta + n - 2) r^{\beta-2} \tag{1}$$

for any complex β if the origin is excluded when the quantities become too singular. It is certainly valid for $\Re\mathfrak{e}(\beta) > 2 - n$ since then both sides of (1) are conventional functions (r^β is conventional if $\Re\mathfrak{e}(\beta) > -n$). It would be convenient if (1) were satisfied by the generalised function r^β. Then $r^{\beta-2}$ could be obtained from r^β and so, by stepping up two at a time in β, a conventional power would be reached eventually (this is the continuation process mentioned in the introduction). Thus $r^{\beta-2}$ could be expressed via generalised derivatives of a conventional function. However, the procedure of stepping up two at a time comes to a halt if either of the factors on the right-hand side of (1) vanishes. This happens, for instance, when $\beta = 2 - n$. It follows that r^{-n} cannot be defined by (1). As a consequence $r^{-n-2}, r^{-n-4}, \ldots$ are undefined as well. Another possible source of failure is $\beta = 0$ but this arises only when $n = 2$ (the stepping up ceases while β is non-zero if $n \geq 3$) and it is known already that r^{-2} is undefined by (1) when $n = 2$. Therefore $\beta = 0$ does not need to be considered further and (1) can be used when $\beta \neq -n - 2k$ ($k = 0, 1, \ldots$).

It is not too surprising that (1) fails for $\beta = 2 - n$ since it can be shown that

$$\nabla^2 r^{2-n} = -\frac{4\pi^{n/2}\delta(\mathbf{x})}{(\frac{1}{2}n - 2)!} \tag{2}$$

which is actually zero when $n = 2$ on account of the factorial being infinite. This shows also, on putting $\beta = \mu + 2 - n$ in (1), that

$$\mathop{\mathrm{Lim}}_{\mu\to 0} \mu r^{\mu-n} = \frac{2\pi^{n/2}\delta(\mathbf{x})}{(\frac{1}{2}n - 1)!}. \tag{3}$$

Apply the operator ∇^2 to (3) k times. Since generalised limits and derivatives can be interchanged (1) gives

$$\mathop{\mathrm{Lim}}_{\mu\to 0} \mu r^{\mu-n-2k} = \frac{\pi^{n/2}(\nabla^2)^k\delta(\mathbf{x})}{(\frac{1}{2}n + k - 1)!k!2^{2k-1}}. \tag{4}$$

The properties (3) and (4) suggest (k always signifies a non-negative integer)

DEFINITION 2.1. *The generalised function r^β ($\beta \neq 2-n-2k$) satisfies*

$$\nabla^2 r^\beta = \beta(\beta+n-2)r^{\beta-2}.$$

The generalised function r^{-n-2k} is defined by

$$r^{-n-2k} = \operatorname*{Lim}_{\mu\to 0}\left\{ r^{\mu-n-2k} - \frac{\pi^{n/2}(\nabla^2)^k\delta(\mathbf{x})}{(\frac{1}{2}n+k-1)!k!2^{2k-1}\mu} \right\}.$$

Obviously, the generalised functions r^β and r^{-n-2k} agree with the conventional functions r^β and r^{-n-2k} respectively whenever $r>0$..

Since $\mathrm{Lim}_{\beta\to\beta_0} r^\beta = r^{\beta_0}$ when $2-n > \Re\mathfrak{e}\beta_0 > -n$, only conventional functions being involved, it follows from (1) that $\mathrm{Lim}_{\beta\to\beta_0} r^\beta = r^{\beta_0}$ so long as $\beta_0 \neq -n-2k$.

The conventional function r^β satisfies

$$\partial_j r^\beta = \beta x_j r^{\beta-2}. \tag{5}$$

Therefore it is valid for $4-n > \Re\mathfrak{e}(\beta) > 2-n$. Apply the operator ∇^2 and use the formula

$$\nabla^2(x_j g) = x_j\nabla^2 g + 2\partial_j g \tag{6}$$

to obtain

$$\beta(\beta+n-4)\{\partial_j r^{\beta-2} - (\beta-2)x_j r^{\beta-4}\} = 0.$$

The factor $\beta+n-4$ is non-zero and β cannot be zero unless $n=3$. Hence (5) holds for $2-n > \Re\mathfrak{e}(\beta) > -n$ with the possible exception of $\beta=-2$ and $n=3$. But then the generalised limit applied to (5) removes the exception. Thereafter repetition of the procedure reduces β by 2 and so (5) holds subject to $\beta \neq 2-n-2k$.

In fact (5) is valid for $\beta = 2-n$ also. From Definition 2.1

$$x_j r^{-n} = \operatorname*{Lim}_{\mu\to 0} x_j r^{\mu-n}$$

since $x_j\delta(\mathbf{x}) = 0$. Invoking (5) on the right-hand side and taking the limit we have

$$\partial_j r^{2-n} = (2-n)x_j r^{-n}$$

which is the same as (5) supplies. *Hence the only restriction on (5) is* $\beta \neq -n-2k$.

Another result concerns the formula

$$r^2.r^\beta = r^{\beta+2} \tag{7}$$

for conventional functions. Take $2-n > \Re\mathfrak{e}\beta > -n$, apply the operator ∇^2 and use

$$\nabla^2(r^2 g) = r^2\nabla^2 g + 4\sum_{j=1}^{n} x_j\partial_j g + 2ng. \tag{8}$$

After taking account of (5) we obtain

$$\beta(\beta+n+2)(r^2.r^{\beta-2} - r^\beta) = 0.$$

Cancellation of the non-zero factors leads to (7) with β reduced by 2. Repetition of the process verifies (7) for $\beta \neq -n-2k$.

The condition $\beta \neq -n-2k$ on (7) can be dropped almost immediately. Since $x_j\delta(\mathbf{x}) = 0$

$$\sum_{j=1}^{n} x_j \partial_j \delta(\mathbf{x}) + n\delta(\mathbf{x}) = 0$$

and then application of (6) repeatedly supplies

$$\sum_{j=1}^{n} x_j \partial_j (\nabla^2)^k \delta(\mathbf{x}) = -(n+2k)(\nabla^2)^k \delta(\mathbf{x}). \tag{9}$$

Moreover $r^2\delta(\mathbf{x}) = 0$ and so (8) and (9) provide

$$0 = (\nabla^2)^k r^2 \delta(\mathbf{x}) = r^2 (\nabla^2)^k \delta(\mathbf{x}) - 2k(n+2k-2)(\nabla^2)^{k-1}\delta(\mathbf{x}). \tag{10}$$

From Definition 2.1 and (7)

$$\begin{aligned} r^2.r^{-n-2k} &= \operatorname*{Lim}_{\mu\to 0} \left\{ r^{\mu-n-2k+2} - \frac{\pi^{n/2} r^2 (\nabla^2)^k \delta(\mathbf{x})}{(\frac{1}{2}n+k-1)!k!2^{2k-1}\mu} \right\} \\ &= r^{-n-2k+2} \end{aligned}$$

by virtue of (10) and Definition 2.1. Thus, (7) *is valid for any complex* β.

Consequently, it has been shown that multiplication of r^β by r^2 produces the same result as when the restriction $r > 0$ is imposed. As pointed out in the introduction this means that the standard rule for the derivative may not hold for all β. Formulae where modification is necessary will be derived now.

Although (7) holds for any complex β both (1) and (5) have to be altered when $\beta = -n-2k$. From Definition 2.1 and (5)

$$\partial_j r^{-n-2k} = \operatorname*{Lim}_{\mu\to 0} \left\{ (\mu-n-2k)x_j r^{\mu-n-2k-2} - \frac{\pi^{n/2}\partial_j (\nabla^2)^k \delta(\mathbf{x})}{(\frac{1}{2}n+k-1)!k!2^{2k-1}\mu} \right\}.$$

The application of (6) to $x_j\delta(\mathbf{x}) = 0$ gives

$$x_j (\nabla^2)^k \delta(\mathbf{x}) = -2k\partial_j (\nabla^2)^{k-1}\delta(\mathbf{x}). \tag{11}$$

Insertion of this result and taking the limit furnishes

$$\partial_j r^{-n-2k} = -(n+2k)x_j r^{-n-2k-2} + \frac{\pi^{n/2} x_j (\nabla^2)^{k+1}\delta(\mathbf{x})}{(\frac{1}{2}n+k)!(k+1)!2^{2k+1}} \tag{12}$$

when benefit is drawn from (4).

It may be verified in a similar manner that (1) is modified to

$$\begin{aligned} \nabla^2 r^{-n-2k} &= (n+2k)(2k+2)r^{-n-2k-2} \\ &\quad - \frac{(n+4k+2)\pi^{n/2}(\nabla^2)^{k+1}\delta(\mathbf{x})}{(\frac{1}{2}n+k)!(k+1)!2^{2k+1}}. \end{aligned} \tag{13}$$

Both (12) and (13) would coincide with their conventional counterparts were it not for the presence of the terms involving $\delta(\mathbf{x})$.

This section has concentrated on certain generalised functions with a singularity at the origin. However, linear changes of variable are permissible and acceptable to the generalised limit so that there is no difficulty in transferring the singularity to a point other than the origin and introducing a dilation. This remark is relevant to later sections also where having the singularity at the origin produces a convenient simplification of the analysis.

3. Logarithmic generalised functions

With the properties of r^β available from the preceding section it is a relatively straightforward matter to introduce logarithmic factors. Start with the case when $\beta \neq -n - 2k$.

DEFINITION 3.1. *The generalised function* $r^\beta \log^m r (\beta \neq -n - 2k)$ *with m a non-negative integer is defined by*

$$r^\beta \log^m r = \frac{\partial^m}{\partial\beta^m} r^\beta.$$

An immediate consequence of this definition is that

$$\nabla^2 r^\beta \log^m r = \nabla^2 \frac{\partial^m}{\partial\beta^m} r^\beta = \frac{\partial^m}{\partial\beta^m} \nabla^2 r^\beta$$

so that, from (1),

$$\begin{aligned} \nabla^2 r^\beta \log^m r &= \frac{\partial^m}{\partial\beta^m} \beta(\beta + n - 2) r^{\beta-2} \\ &= \beta(\beta + n - 2) r^{\beta-2} \log^m r + m(2\beta + n - 2) r^{\beta-2} \log^{m-1} r \\ &\quad + m(m-1) r^{\beta-2} \log^{m-2} r \end{aligned} \tag{14}$$

provided that $\beta \neq 2 - n - 2k$.

By virtue of (5)

$$\partial_j (r^\beta \log^m r) = (\beta r^{\beta-2} \log^m r + m r^{\beta-2} \log^{m-1} r) x_j \tag{15}$$

when $\beta \neq -n - 2k$ while (7) provides

$$r^2 . r^\beta \log^m r = r^{\beta+2} \log^m r. \tag{16}$$

For the exceptional values of β take advantage of Definition 2.1 and introduce

DEFINITION 3.2. *The generalised function* $r^{-n-2k} \log^m r$ *is defined by*

$$r^{-n-2k} \log^m r = \operatorname*{Lim}_{\mu \to 0} \frac{\partial^m}{\partial\mu^m} \left\{ r^{\mu-n-2k} - \frac{\pi^{n/2} (\nabla^2)^k \delta(\mathbf{x})}{(\frac{1}{2}n + k - 1)! k! 2^{2k-1} \mu} . \right\}$$

That (16) holds for any β follows at once from Definition 3.2, (16) and (10).

Again the standard effect of multiplying by r^2 has been retained at the expense of altering the derivative for some values of β.

From Definition 3.2, (15) and (11)

$$\partial_j(r^{-n-2k}\log^m r) = -(n+2k)x_j r^{-n-2k-2}\log^m r + m x_j r^{-n-2k-2}\log^{m-1} r + \frac{\pi^{n/2}x_j(\nabla^2)^{k+1}\delta(\mathbf{x})}{(-m)!(\frac{1}{2}n+k)!(k+1)!2^{2k+1}}. \tag{17}$$

Notice that (17) agrees with (15) for $m \geq 1$. Also

$$\begin{aligned}\nabla^2(r^{-n-2k}\log^m r) &= (n+2k)(2k+2)r^{-n-2k-2}\log^m r + m(m-1)r^{-n-2k-2}\log^{m-2} r \\ &\quad - m(n+4k+2)r^{-n-2k-2}\log^{m-1} r \\ &\quad + \left\{\frac{m}{(1-m)!} - \frac{n+4k+2}{(-m)!}\right\}\frac{\pi^{n/2}(\nabla^2)^{k+1}\delta(\mathbf{x})}{(\frac{1}{2}n+k)!(k+1)!2^{2k+1}}.\end{aligned} \tag{18}$$

Formula (18) is the same as (14) when $m \geq 2$.

4. Fourier transforms

In this section are derived the Fourier transforms of the singular generalised functions considered in the preceding two sections. The simplest case will be dealt with first. The symbol $\tilde{\nabla}^2$ denotes the Laplacian in $\boldsymbol{\alpha}$-space.

THEOREM 4.1. *The Fourier transform of* r^{2k} *is* $(-1)^k(2\pi)^n(\tilde{\nabla}^2)^k\delta(\boldsymbol{\alpha})$.

Proof. The Fourier transform of $\delta(\mathbf{x})$ is 1. By the inversion theorem the Fourier transform of 1 is $(2\pi)^n\delta(\boldsymbol{\alpha})$. Since the Fourier transform of $x_j g(\mathbf{x})$ is $\mathrm{i}\tilde{\partial}_j G(\boldsymbol{\alpha})$, where the partial derivative $\tilde{\partial}_j$ is with respect to α_j, the theorem follows at once. ∎

The corresponding theorem for r^β is

THEOREM 4.2. *If* $\beta \neq 2k$ *and* $\beta \neq -n-2k$ *the Fourier transform of* r^β *is*

$$\frac{(\frac{1}{2}\beta+\frac{1}{2}n-1)!}{(-\frac{1}{2}\beta-1)!}2^{\beta+n}\pi^{n/2}\alpha^{-\beta-n}.$$

Proof. Suppose, firstly, that $0 > \Re\mathfrak{e}\beta > -n$. Then $\mathrm{Lim}_{\mu\to 0}\mathrm{e}^{-\mu r}r^\beta = r^\beta$ since this is a conventional result. In addition, with $\mu > 0$, $\mathrm{e}^{-\mu r}r^\beta$ is absolutely integrable. Therefore its Fourier transform can be evaluated in spherical polar coordinates and

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\mu r-\mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{x}}r^\beta\,\mathrm{d}\mathbf{x} = (2\pi)^{n/2}\int_0^\infty \mathrm{e}^{-\mu r}r^{\beta+n-1}\frac{J_{\frac{1}{2}n-1}(\alpha r)}{(\alpha r)^{\frac{1}{2}n-1}}\,\mathrm{d}r$$

where $J_\nu(z)$ is the standard Bessel function. The integral on the right is available in Watson [4, p. 385] and gives

$$\frac{(\beta+n-1)!2\pi^{n/2}}{(\frac{1}{2}n-1)!(\mu^2+\alpha^2)^{(\beta+n)/2}}F(\tfrac{1}{2}\beta+\tfrac{1}{2}n, -\tfrac{1}{2}\beta-\tfrac{1}{2}, \tfrac{1}{2}n, \frac{\alpha^2}{\mu^2+\alpha^2})$$

with F the usual hypergeometric function. When $\mu \to 0$ a standard formula for the hypergeometric function supplies

$$F(\tfrac{1}{2}\beta+\tfrac{1}{2}n, -\tfrac{1}{2}\beta-\tfrac{1}{2}, \tfrac{1}{2}n, 1) = \frac{(\frac{1}{2}n-1)!\pi^{1/2}}{(-\frac{1}{2}\beta-1)!(\frac{1}{2}n+\frac{1}{2}\beta-\frac{1}{2})!}.$$

Also $0 > \Re\mathfrak{e}(-\beta - n) > -n$ so that $(\mu^2 + \alpha^2)^{-(\beta+n)/2}$ is conventional and

$$\operatorname*{Lim}_{\mu\to 0}(\mu^2 + \alpha^2)^{-(\beta+n)/2} = \alpha^{-\beta-n}.$$

Insertion of these limits and use of the formula $(2z)! = z!(z - \frac{1}{2})!2^{2z}/\pi^{1/2}$ reproduces the transform stated in the theorem. Hence the theorem is proved for $0 > \Re\mathfrak{e}\beta > -n$.

The transform of $\partial_j g(\mathbf{x})$ is $\mathrm{i}\alpha_j G(\boldsymbol{\alpha})$. Accordingly, the transform of $\nabla^2 r^\beta$ or $\beta(\beta+n-2)r^{\beta-2}$ is $-\alpha^2$ times the previous result. Since $\beta \neq 0$ and $\beta = 2 - n$ would make $\beta - 2$ an excluded value we see that the transform of $r^{\beta-2}$ is the same as that for r^β with $\beta - 2$ for β. Repetition of this argument reveals that the theorem is true for $\Re\mathfrak{e}\beta < 0$ apart from the excluded values.

By the Fourier inversion theorem and what has been proved already

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\frac{(\frac{1}{2}\gamma + \frac{1}{2}n - 1)!}{(-\frac{1}{2}\gamma - 1)!}2^{\gamma+n}\pi^{n/2}\alpha^{-\gamma-n}\mathrm{e}^{\mathrm{i}d\boldsymbol{\alpha}\cdot\mathbf{x}}\boldsymbol{\alpha} = r^\gamma$$

for $\Re\mathfrak{e}\gamma < 0$ and $\gamma \neq -n - 2k$. Put $\gamma + n = -\beta$ so that $\Re\mathfrak{e}\beta > -n$ and $\beta \neq 2k$. Then the theorem is confirmed for this range of β.
The proof is complete. ■

Observe that Theorem 4.1 can be recovered from Theorem 4.2 by allowing $\beta \to 2k$. This is confirmed by making the substitution $-\pi/(-\frac{1}{2}\beta - 1)! = (\frac{1}{2}\beta)!\sin\frac{1}{2}\beta\pi$ and calling on (4). Since $\operatorname{Lim}_{\beta\to 2k} r^\beta = r^{2k}$ this is actually a check on consistency.

The remaining exceptional values are covered by

THEOREM 4.3. *The Fourier transform of r^{-n-2k} is*

$$\frac{(-1)^k\pi^{n/2}\alpha^{2k}}{(\frac{1}{2}n + k - 1)!k!2^{2k-1}}\left\{\tfrac{1}{2}\psi(\tfrac{1}{2}n + k - 1) + \tfrac{1}{2}\psi(k) - \log\tfrac{1}{2}\alpha\right\}$$

where $\psi(z) = z!'/z!$, the prime indicating a derivative with respect to the argument.

Proof. By Definition 2.1 and Theorem 4.2 the Fourier transform of r^{-n-2k} is

$$\operatorname*{Lim}_{\mu\to 0}\left\{\frac{(\frac{1}{2}\mu - k - 1)!}{(-\frac{1}{2}\mu + \frac{1}{2}n + k - 1)!}2^{\mu-2k}\pi^{n/2}\alpha^{-\mu+2k} - \frac{(-1)^k\pi^{n/2}\alpha^{2k}}{(\frac{1}{2}n + k - 1)!k!2^{2k-1}\mu}\right\}.$$

The replacement

$$(\tfrac{1}{2}\mu - k - 1)! = \frac{(-1)^k}{(k - \frac{1}{2}\mu)!\sin\frac{1}{2}\mu\pi}$$

leads to

$$(-1)^k\pi^{n/2}\frac{\alpha^{2k}}{2^{2k-1}}\left[\frac{\partial}{\partial\mu}\frac{2^\mu\alpha^{-\mu}}{(k - \frac{1}{2}\mu)!(\frac{1}{2}n - \frac{1}{2}\mu + k - 1)!}\right]_{\mu=0}$$

which gives the formula stated in the theorem. The proof is finished. ■

With regard to logarithms we have

THEOREM 4.4. *If* $\beta \neq 2k$ *and* $\beta \neq -n-2k$ *the Fourier transform of* $r^\beta \log^m r$ *is*

$$\frac{(\frac{1}{2}\beta + \frac{1}{2}n - 1)!}{(-\frac{1}{2}\beta - 1)!} 2^{\beta+n} \pi^{n/2} \alpha^{-\beta-n} \phi_m(\beta)$$

where $\phi_1(\beta) = \frac{1}{2}\psi(\frac{1}{2}\beta + \frac{1}{2}n - 1) + \frac{1}{2}\psi(-\frac{1}{2}\beta - 1) - \log \frac{1}{2}\alpha$ *and, for* $m > 1$,

$$\phi_m(\beta) = \phi_1(\beta)\phi_{m-1}(\beta) + \phi'_{m-1}(\beta).$$

Proof. The theorem is an immediate consequence of Definition 3.1 and Theorem 4.2. ■

When $\beta = -n-2k$ the pertinent theorem is

THEOREM 4.5. *The Fourier transform of* $r^{-n-2k} \log^m r$ *is*

$$\frac{(-1)^k \pi^{n/2} \alpha^{2k} \tau_{m+1}(0)}{(\frac{1}{2}n + k - 1)!k!(m+1)2^{2k-1}}$$

where

$$\tau_1(\mu) = \tfrac{1}{2}\psi(k - \tfrac{1}{2}\mu) + \tfrac{1}{2}\psi(\tfrac{1}{2}n + k - \tfrac{1}{2}\mu - 1) + \tfrac{1}{2}\psi(\mu) - \tfrac{1}{2}\psi(-\tfrac{1}{2}\mu) - \log \tfrac{1}{2}\alpha$$

and for $m > 1$

$$\tau_m(\mu) = \tau_1(\mu)\tau_{m-1}(\mu) + \tau'_{m-1}(\mu).$$

There is an alternative expression for $\tau_m(\mu)$ which separates out the powers of $\log \alpha$, namely

$$\tau_m(\mu) = \sum_{p=0}^{m} \frac{m!(-1)^{p+m}}{p!(m-p)!} \chi_p(\mu) \log^{m-p} \alpha \tag{19}$$

where $\chi_0(\mu) = 1$, $\chi_m(\mu) = \chi(\mu)\chi_{m-1}(\mu) + \chi'_{m-1}(\mu)$ $(m \geq 1)$ and

$$\chi(\mu) = \tfrac{1}{2}\psi(k - \tfrac{1}{2}\mu) + \tfrac{1}{2}\psi(\tfrac{1}{2}n + k - \tfrac{1}{2}\mu - 1) + \tfrac{1}{2}\psi(\mu) - \tfrac{1}{2}\psi(-\tfrac{1}{2}\mu) + \log 2. \tag{20}$$

Since $\tau_1(\mu) = \chi(\mu) - \log \alpha$, Equation (19) is verified for $m = 1$. Then induction confirms (19) for general m.

Proof. According to Definition 3.2 and Theorem 4.2 the Fourier transform of $r^{-n-2k} \log^m r$ is

$$\operatorname*{Lim}_{\mu \to 0} \frac{\partial^m}{\partial \mu^m} \left\{ \frac{(\frac{1}{2}\mu - k - 1)! 2^{\mu - 2k}}{(-\frac{1}{2}\mu + \frac{1}{2}n + k - 1)!} \pi^{n/2} \alpha^{2k-\mu} - \frac{(-1)^k \pi^{n/2} \alpha^{2k}}{(\frac{1}{2}n + k - 1)!k!2^{2k-1}\mu} \right\}.$$

Rewrite this expression as

$$(-1)^k \pi^{n/2} \frac{\alpha^{2k}}{2^{2k-1}} \operatorname*{Lim}_{\mu \to 0} \frac{\partial^m}{\partial \mu^m} \frac{1}{\mu} \{f(\mu) - f(0)\}$$

where

$$f(\mu) = \frac{(\frac{1}{2}\mu)!(-\frac{1}{2}\mu)!2^\mu \alpha^{-\mu}}{(k - \frac{1}{2}\mu)!(\frac{1}{2}n + k - 1 - \frac{1}{2}\mu)!}.$$

Consequently, the Fourier transform is

$$(-1)^k \pi^{n/2} \frac{\alpha^{2k}}{2^{2k-1}} \frac{f^{(m+1)}(0)}{m+1}.$$

Since $f^{(m)}(\mu) = f(\mu)\tau_m(\mu)$ the proof is finished. ■

The final exceptional value of β is dealt with in

THEOREM 4.6. *The Fourier transform of* $r^{2k}\log^m r$ *is*

$$(\tfrac{1}{2}n+k-1)!k!(-1)^{k+m}2^{2k+n-1}\pi^{n/2}\sum_{p=0}^{m-1}\frac{m!\rho_p(0)}{p!(m-1-p)!}\alpha^{-n-2k}\log^{m-1-p}\alpha$$
$$+(-1)^{k+m}(2\pi)^n\rho_m(0)(\tilde{\nabla}^2)^k\delta(\boldsymbol{\alpha})$$

where $\rho_0(\mu)=1$, $\rho_m(\mu)=\rho(\mu)\rho_{m-1}(\mu)+\rho'_{m-1}(\mu)$ $(m\geq 1)$ *and* $\rho(\mu)=-\chi(\mu)$.

Proof. Either use $\mathrm{Lim}_{\beta\to 2k}r^\beta\log^m r=r^{2k}\log^m r$ and Theorem 4.2 or take the Fourier inverse of Theorem 4.5 and employ (19), though this involves more manipulation.
The proof is ended. ■

5. Another type of singularity

The generalised functions r^β and $r^\beta\log^m r$ have no singularity other than at the origin. With this attribute they resemble the singular generalised functions in one dimension. But generalised functions in R_n can have singularities on hypersurfaces. Discussion of generalised functions which have singularities on conical-like boundaries is contained in Jones [3, Chapter 8]. Therefore the case of a singularity at a conical point has been dealt with fully, so here some generalised functions which have a singularity on a smooth boundary are examined. To illustrate this feature we will consider powers of r^2-1 which can have singularities on the surface of the unit sphere. The singularities can be transferred to other places and other shapes by means of linear mapping. Since it is easy to carry out the transference only details for r^2-1 will be given.

The non-negative integer powers of r^2-1 are covered by the preceding sections via the binomial theorem. Negative integer powers require a definition since they are non-integrable. In the following m denotes a positive integer.

DEFINITION 5.1. *The generalised function* $(r^2-1)^{-m}$ *is defined by*

$$(r^2-1)^{-1} = \frac{1}{2}\sum_{j=1}^{n}x_j\partial_j\log\left|r^2-1\right|-1,$$
$$(r^2-1)^{-m} = \frac{-1}{2(m-1)}\sum_{j=1}^{n}x_j\partial_j(r^2-1)^{-m+1}-(r^2-1)^{-m+1}\quad(m\geq 2).$$

The function $\log|r^2-1|$ is conventional and integrable. Also x_j is fairly good so that $(r^2-1)^{-m}$ is a well-defined generalised function which agrees with the conventional version when $r^2\neq 1$.

The function $(r^2-1)\log|r^2-1|$ is conventional and r^2-1 is fairly good. Therefore the derivative may be calculated either by conventional or by generalised means. On equating the two $(r^2-1).(r^2-1)^{-1}=1$. The more general result

$$(r^2-1).(r^2-1)^{-m}=(r^2-1)^{-m+1} \tag{21}$$

then follows from the definition without difficulty through induction. The conventional rule for multiplication has been preserved.

For more general powers of $r^2 - 1$ the change in sign as r passes through 1 has to be circumvented. The device is to separate consideration of $r^2 > 1$ and $r^2 < 1$. When $\Re\mathfrak{e}(\beta) > -1$ let

$$(r^2-1)_+^\beta = (r^2-1)^\beta H(r^2-1)$$

where $H(x)$ is the Heaviside step function which is 1 for $x > 0$ and 0 for $x < 0$. In other words $(r^2-1)_+^\beta$ is zero for $r^2 < 1$.
Similarly

$$(r^2-1)_-^\beta = (1-r^2)^\beta H(1-r^2)$$

is zero for $r^2 > 1$.

Knowing $(r^2-1)_+^\beta$ as a conventional function for $\Re\mathfrak{e}\beta > -1$ we go to lower values of β by generalised derivatives in a similar manner to that for r^β.

DEFINITION 5.2. *When* $\beta \neq -m$ *the generalised functions* $(r^2-1)_+^\beta$ *and* $(r^2-1)_-^\beta$ *are given by*

$$(r^2-1)_+^\beta = \frac{1}{2(\beta+1)}\sum_{j=1}^n x_j\partial_j(r^2-1)_+^{\beta+1} - (r^2-1)_+^{\beta+1},$$

$$(r^2-1)_-^\beta = (r^2-1)_-^{\beta+1} - \frac{1}{2(\beta+1)}\sum_{j=1}^n x_j\partial_j(r^2-1)_-^{\beta+1}.$$

If it be assumed that $(r^2-1).(r^2-1)_+^{\beta+1} = (r^2-1)_+^{\beta+2}$, which is certainly true when $\Re\mathfrak{e}(\beta)$ is large enough,

$$\begin{aligned}\sum_{j=1}^n x_j\partial_j(r^2-1)_+^{\beta+2} &= \sum_{j=1}^n x_j\partial_j(r^2-1).(r^2-1)_+^{\beta+1}\\ &= 2r^2(r^2-1)_+^{\beta+1} + (r^2-1)\sum_{j=1}^n x_j\partial_j(r^2-1)_+^{\beta+1}.\end{aligned}$$

Replace the derivatives by means of Definition 5.2 to obtain

$$2(\beta+1)(r^2-1)_+^{\beta+1} = 2(\beta+1)(r^2-1).(r^2-1)_+^\beta.$$

Since $\beta \neq -1$

$$(r^2-1).(r^2-1)_+^\beta = (r^2-1)_+^{\beta+1}. \tag{22}$$

Induction then verifies (22) for all $\beta \neq -m$.

In like manner

$$(r^2-1).(r^2-1)_-^\beta = -(r^2-1)_-^{\beta+1} \tag{23}$$

when $\beta \neq -m$.

The missing values of β are covered by

DEFINITION 5.3. *The generalised functions* $(r^2-1)_+^{-m}$ *and* $(r^2-1)_-^{-m}$ *are defined by*

$$(r^2-1)_+^{-m} = \operatorname*{Lim}_{\mu\to 0}\left\{(r^2-1)_+^{\mu-m} - \frac{(-1)^{m-1}\delta^{(m-1)}(r^2-1)}{(m-1)!\mu}\right\},$$

$$(r^2-1)_-^{-m} = \operatorname*{Lim}_{\mu\to 0}\left\{(r^2-1)_-^{\mu-m} - \frac{\delta^{(m-1)}(r^2-1)}{(m-1)!\mu}\right\}.$$

It is necessary to check that the generalised limits exist. It will be sufficient to indicate the method for $m=1$. Now, from Definition 5.2,

$$(r^2-1)_+^{\mu-1} = \frac{1}{2\mu}\sum_{j=1}^{n} x_j\partial_j(r^2-1)_+^{\mu} - (r^2-1)_+^{\mu} \tag{24}$$

The conventional function $(r^2-1)_+^{\mu}$ can be expanded in powers of μ so that

$$(r^2-1)_+^{\mu} = H(r^2-1)\left\{1+\mu\log\left|r^2-1\right| + O(\mu^2)\right\}.$$

Then the right-hand side of (24) becomes

$$\frac{1}{\mu}\delta(r^2-1) + \frac{1}{2}\sum_{j=1}^{n} x_j\partial_j\left\{H(r^2-1)\log\left|r^2-1\right|\right\} - H(r^2-1) + o(1)$$

as $\mu\to 0$. Hence

$$(r^2-1)_+^{-1} = \frac{1}{2}\sum_{j=1}^{n} x_j\partial_j\left\{H(r^2-1)\log\left|r^2-1\right|\right\} - H(r^2-1). \tag{25}$$

Similarly

$$(r^2-1)_-^{-1} = H(1-r^2) - \frac{1}{2}\sum_{j=1}^{n} x_j\partial_j\left\{H(1-r^2)\log\left|r^2-1\right|\right\}. \tag{26}$$

By combining (25) and (26) we have, since $H(x)+H(-x)=1$,

$$(r^2-1)_+^{-1} - (r^2-1)_-^{-1} = \frac{1}{2}\sum_{j=1}^{n} x_j\partial_j\log\left|r^2-1\right| - 1 = (r^2-1)^{-1} \tag{27}$$

on quoting Definition 5.1.

Since, by (22),

$$(r^2-1).(r^2-1)_+^{-m} = \operatorname*{Lim}_{\mu\to 0}\left\{(r^2-1)_+^{\mu-m-1} - \frac{(-1)^{m-1}(r^2-1)\delta^{(m-1)}(r^2-1)}{(m-1)!\mu}\right\}$$

and

$$(r^2-1)\delta^{(k)}(r^2-1) = -k\delta^{(k-1)}(r^2-1) \tag{28}$$

it follows that (22) holds when $\beta=-m$. Similarly, (23) is valid also when $\beta=-m$. Consequently, *the usual rule for multiplication by* r^2-1 *has been adhered to.* The exceptional cases for the derivative will be dealt with now.

By applying ∂_j in Definition 5.3 and calling on Definition 5.2 we find, after use of (28),

$$\frac{1}{2}\sum_{j=1}^{n} x_j \partial_j (r^2-1)_+^{-m} = -m\left\{(r^2-1)_+^{-m-1} + (r^2-1)_+^{-m}\right\} + \frac{(-1)^m}{m!}\delta^{(m)}(r^2-1) + \frac{(-1)^{m-1}}{(m-1)!}\delta^{(m-1)}(r^2-1), \tag{29}$$

$$\frac{1}{2}\sum_{j=1}^{n} x_j \partial_j (r^2-1)_-^{-m} = -m\left\{(r^2-1)_-^{-m} - (r^2-1)_-^{-m-1}\right\} + \frac{1}{(m-1)!}\delta^{(m-1)}(r^2-1) - \frac{1}{m!}\delta^{(m)}(r^2-1). \tag{30}$$

From (29) and (30)

$$\frac{1}{2}\sum_{j=1}^{n} x_j \partial_j \left\{(r^2-1)_+^{-m} + \mathrm{e}^{-m\pi\mathrm{i}}(r^2-1)_-^{-m}\right\}$$
$$= -m\left\{(r^2-1)_+^{-m} + \mathrm{e}^{-m\pi\mathrm{i}}(r^2-1)_-^{-m} + (r^2-1)_+^{-m-1} + \mathrm{e}^{-(m+1)\pi\mathrm{i}}(r^2-1)_-^{-m-1}\right\}.$$

If now

$$(r^2-1)_+^{-m} + \mathrm{e}^{-m\pi\mathrm{i}}(r^2-1)_-^{-m} = (r^2-1)^{-m}, \tag{31}$$

which is true for $m = 1$ by (27), Definition 5.1 gives

$$m(r^2-1)^{-m-1} = m\left\{(r^2-1)_+^{-m-1} + \mathrm{e}^{-(m+1)\pi\mathrm{i}}(r^2-1)_-^{-m-1}\right\}.$$

But, since $m \neq 0$, this is the same as (31) with m replaced by $m+1$. Hence (31) is verified for all m by induction.

The function $(r^2 - 1 + \mathrm{i}\varepsilon)^\beta$, with $\varepsilon > 0$, is well-defined for all values of r so long as the phase of the complex number $r^2 - 1 + \mathrm{i}\varepsilon$ is specified. It will be taken to lie in the range $(0, \pi)$. Also the phase of $r^2 - 1 - \mathrm{i}\varepsilon$ will be restricted to $(-\pi, 0)$. With these conventions on the phase we introduce

DEFINITION 5.4. *The generalised functions* $(r^2 - 1 \pm \mathrm{i}0)^\beta$ *are defined by*

$$(r^2-1\pm\mathrm{i}0)^\beta = \operatorname*{Lim}_{\varepsilon\to+0}(r^2-1\pm\mathrm{i}\varepsilon)^\beta.$$

Since

$$\frac{1}{2\beta}\sum_{j=1}^{n} x_j \partial_j (r^2-1\pm\mathrm{i}\varepsilon)^\beta - (r^2-1\pm\mathrm{i}\varepsilon)^\beta = (1\mp\mathrm{i}\varepsilon)(r^2-1\pm\mathrm{i}\varepsilon)^{\beta-1}$$

it follows that

$$\frac{1}{2\beta}\sum_{j=1}^{n} x_j \partial_j (r^2-1\pm\mathrm{i}0)^\beta - (r^2-1\pm\mathrm{i}0)^\beta = (r^2-1\pm\mathrm{i}0)^{\beta-1}. \tag{32}$$

It is obvious that, when $\Re\mathfrak{e}(\beta) > -1$,

$$(r^2-1\pm\mathrm{i}0)^\beta = (r^2-1)_+^\beta + \mathrm{e}^{\pm\beta\pi\mathrm{i}}(r^2-1)_-^\beta. \tag{33}$$

Apply the operator $\sum_{j=1}^{n} x_j \partial_j$ and invoke (32). Then Definition 5.2 reproduces (33) with β reduced by 1. Hence (33) holds for all β except possibly for β a negative integer. On account of (22) and (23)

$$(r^2 - 1).(r^2 - 1 \pm \mathrm{i}0)^{\beta} = (r^2 - 1 \pm \mathrm{i}0)^{\beta+1} \tag{34}$$

and the standard rule for multiplication is applicable.

Now

$$\frac{1}{2}\sum_{j=1}^{n} x_j \partial_j \log(r^2 - 1 \pm \mathrm{i}\varepsilon) - 1 = (1 \mp \mathrm{i}\varepsilon)(r^2 - 1 \pm \mathrm{i}\varepsilon)^{-1}$$

so that

$$\begin{aligned}(r^2 - 1 \pm \mathrm{i}0)^{-1} &= \operatorname*{Lim}_{\varepsilon \to +0} \frac{1}{2}\sum_{j=1}^{n} x_j \partial_j \log(r^2 - 1 \pm \mathrm{i}\varepsilon) - 1 \\ &= \frac{1}{2}\sum_{j=1}^{n} x_j \partial_j \left\{\log\left|r^2 - 1\right| \pm \pi \mathrm{i} H(1 - r^2)\right\} - 1 \\ &= (r^2 - 1)^{-1} \mp \pi \mathrm{i}\delta(r^2 - 1)\end{aligned}$$

from Definition 5.1. More generally

$$(r^2 - 1 \pm \mathrm{i}0)^{-m} = (r^2 - 1)^{-m} \mp \frac{(-1)^{m-1}\pi \mathrm{i}}{(m-1)!}\delta^{(m-1)}(r^2 - 1). \tag{35}$$

One consequence of (35) is that (32) *continues to hold when* $\beta = -m$, as may be confirmed via Definition 5.1 and (28). Another inference is that (*34*) *is valid when* $\beta = -m$.

Another property is of interest. From (33)

$$\operatorname*{Lim}_{\mu \to 0}(r^2 - 1 \pm \mathrm{i}0)^{\mu - m} = \operatorname*{Lim}_{\mu \to 0}\left\{(r^2 - 1)_+^{\mu - m} + \mathrm{e}^{\pm(\mu - m)\pi \mathrm{i}}(r^2 - 1)_-^{\mu - m}\right\}.$$

On substitution from Definition 5.3 the right-hand side becomes

$$(r^2 - 1)_+^{-m} + \mathrm{e}^{\mp m\pi \mathrm{i}}(r^2 - 1)_-^{-m} \mp \frac{(-1)^{m-1}\pi \mathrm{i}}{(m-1)!}\delta^{(m-1)}(r^2 - 1).$$

The use of (31) and (35) leads to

$$\operatorname*{Lim}_{\mu \to 0}(r^2 - 1 \pm \mathrm{i}0)^{\mu - m} = (r^2 - 1 \pm \mathrm{i}0)^{-m}. \tag{36}$$

Logarithmic factors can be introduced in a similar manner to that of Section 3 except for $(r^2 - 1)^{-m}$ which has a different kind of definition. For such powers the appropriate approach is

$$(r^2 - 1)^{-1}\log^p\left|r^2 - 1\right| = \frac{1}{2(p+1)}\sum_{j=1}^{n} x_j \partial_j \log^{p+1}\left|r^2 - 1\right| - \log^p\left|r^2 - 1\right|.$$

Lower powers of $r^2 - 1$ can be deduced from

$$\frac{1}{2}\sum_{j=1}^{n} x_j \partial_j (r^2-1)^{1-m} \log^p \left|r^2-1\right| =$$
$$(1-m)\left\{(r^2-1)^{1-m}\log^p\left|r^2-1\right| + (r^2-1)^{-m}\log^p\left|r^2-1\right|\right\}$$
$$+\, p\left\{(r^2-1)^{1-m}\log^{p-1}\left|r^2-1\right| + (r^2-1)^{-m}\log^{p-1}\left|r^2-1\right|\right\}.$$

The more explicit

$$(r^2-1)^{-m}\log^p\left|r^2-1\right| =$$
$$-\sum_{q=0}^{p}\frac{p!}{(p-q)!2(m-1)^{q+1}}\sum_{j=1}^{n} x_j\partial_j(r^2-1)^{1-m}\log^{p-q}\left|r^2-1\right| - (r^2-1)^{1-m}\log^p\left|r^2-1\right|$$

for $m > 1$ may be preferred. Standard multiplication still applies *i.e.*

$$(r^2-1).(r^2-1)^{-m}\log^p\left|r^2-1\right| = (r^2-1)^{1-m}\log^p\left|r^2-1\right|.$$

The analogous formulae for $(r^2-1)_\pm^\beta \log^p\left|r^2-1\right|$ are

$$\frac{1}{2}\sum_{j=1}^{n} x_j\partial_j(r^2-1)_+^{\beta+1}\log^p\left|r^2-1\right| =$$
$$(\beta+1)\left\{(r^2-1)_+^{\beta+1}\log^p\left|r^2-1\right| + (r^2-1)_+^{\beta}\log^p\left|r^2-1\right|\right\}$$
$$+\, p\left\{(r^2-1)_+^{\beta+1}\log^{p-1}\left|r^2-1\right| + (r^2-1)_+^{\beta}\log^{p-1}\left|r^2-1\right|\right\}$$

and

$$\frac{1}{2}\sum_{j=1}^{n} x_j\partial_j(r^2-1)_-^{\beta+1}\log^p\left|r^2-1\right| =$$
$$(\beta+1)\left\{(r^2-1)_-^{\beta+1}\log^p\left|r^2-1\right| - (r^2-1)_-^{\beta}\log^p\left|r^2-1\right|\right\}$$
$$+\, p\left\{(r^2-1)_-^{\beta+1}\log^{p-1}\left|r^2-1\right| - (r^2-1)_-^{\beta}\log^{p-1}\left|r^2-1\right|\right\}$$

provided that $\beta + 1$ is not a negative integer. The same formulae hold when $\beta + 1$ is a negative integer so long as p is positive. Also valid are

$$(r^2-1).(r^2-1)_\pm^\beta\log^p\left|r^2-1\right| = \pm(r^2-1)_\pm^{\beta+1}\log^p\left|r^2-1\right|$$

for any β and

$$(r^2-1)_+^{-m}\log^p\left|r^2-1\right| + e^{-m\pi i}(r^2-1)_-^{-m}\log^p\left|r^2-1\right| = (r^2-1)^{-m}\log^p\left|r^2-1\right|.$$

Consideration of the Founer transforms of the generalised functions introduced in this section is deferred to the next section.

6. Bessel functions

Bessel functions occur in many places in applied mathematics so that it is of interest to consider their properties as generalised functions. In view of the great variety of Bessel funtions a full discussion would be prohibitively long. So our investigation will be limited to one variant but that should be sufficient to indicate how others can be handled.

DEFINITION 6.1. *The generalised function $r^\nu J_\nu(r)$ is defined by*

$$r^\nu J_\nu(r) = \sum_{m=0}^{\infty} \frac{(-1)^m r^{2\nu+2m}}{m!(\nu+m)!2^{\nu+2m}}.$$

When $\Re\mathfrak{e}(\nu) > -n/2$ the definition makes the generalised function the same as the conventional function $r^\nu J_\nu(r)$. For other values of ν removal of a finite number of terms leaves a conventional convergent series and so the whole series is generally convergent. Thus $r^\nu J_\nu(r)$ is a well-defined generalised function. The fact that the series is generally convergent means that derivatives and Fourier transforms can be taken term-by-term when desired.

It follows from Definition 2.1 that, if $\nu \neq 1 - n/2 - k$,

$$(\nabla^2 + 1) r^\nu J_\nu(r) = (2\nu + n - 2) r^{\nu-1} J_{\nu-1}(r). \tag{37}$$

On the other hand, (2) gives

$$(\nabla^2 + 1) r^{1-n/2} J_{1-n/2}(r) = 2^{n/2+1} \pi^{n/2+1} \delta(\boldsymbol{x}) \sin \tfrac{1}{2} n\pi, \tag{38}$$

whereas (13) supplies

$$\begin{aligned}(\nabla^2 + 1) r^{-n/2-k} J_{-n/2-k}(r) = \\ &-2(k+1) r^{-n/2-k-1} J_{-n/2-k-1}(r) \\ &+ (-1)^{k+1} 2^{n/2-k-1} \pi^{n/2-1} \sin \tfrac{1}{2} n\pi \sum_{m=0}^{k+1} \frac{(n+4k-4m+2)(\nabla^2)^{k-m+1}\delta(\boldsymbol{x})}{m!(k+1-m)!(n/2+k-m)}.\end{aligned} \tag{39}$$

Likewise, we obtain via (5)

$$\sum_{j=1}^{n} x_j \partial_j r^\nu J_\nu(r) = r^2 . r^{\nu-1} J_{\nu-1}(r) \tag{40}$$

if $\nu \neq -n/2 - k$. For $\nu = -n/2 - k$ we have, from (12) and (10),

$$\begin{aligned}\sum_{j=1}^{n} x_j \partial_j r^{-n/2-k} J_{-n/2-k}(r) &= r^2 . r^{-n/2-k-1} J_{-n/2-k-1}(r) \\ &+ (-1)^k 2^{n/2-k+1} \pi^{n/2-1} \sin \tfrac{1}{2} n\pi \sum_{m=0}^{k} \frac{(\nabla^2)^{k-m}\delta(\boldsymbol{x})}{m!(k-m)!}.\end{aligned} \tag{41}$$

By combining (37) and (40) we have, for $\nu \neq -n/2 - k$,

$$r^2 (\nabla^2 + 1) r^\nu J_\nu(r) + (2 - n - 2\nu) \sum_{j=1}^{n} x_j \partial_j r^\nu J_\nu(r) = 0, \tag{42}$$

which is the same as the conventional result. Also (39), (41) and (10) provide

$$\begin{aligned}&r^2 (\nabla^2 + 1) r^{-n/2-k} J_{-n/2-k}(r) + 2(k+1) \sum_{j=1}^{n} x_j \partial_j r^{-n/2-k} J_{-n/2-k}(r) \\ &= (-1)^{k+1} 2^{n/2-k+1} \pi^{n/2-1} \sin \tfrac{1}{2} n\pi \sum_{m=0}^{k} \frac{n+2k-4m}{m!(k-m)!} (\nabla^2)^{k-m} \delta(\boldsymbol{x}).\end{aligned} \tag{43}$$

These relations reveal that $r^\nu J_\nu(r)$ is capable of constructing fundamental solutions of certain differential equations when n is odd.

Another formmula which can be useful is

$$\sum_{j=1}^{n} x_j \partial_j r^\nu J_\nu(r) - 2\nu r^\nu J_\nu(r) = -r^{\nu+1} J_{\nu+1}(r) \tag{44}$$

when $\nu \neq -n/2 - k$.

In contrast to $r^\nu J_\nu$ the function $r^{-\nu} J_\nu$ is the same whether regarded as generalised or conventional. It has the representation as a series

$$r^{-\nu} J_\nu(r) = \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m}}{m!(\nu+m)!2^{\nu+2m}}. \tag{45}$$

The Bessel function $r^\nu Y_\nu(r)$ can be defined now by

$$r^\nu Y_\nu(r) = \{r^\nu J_\nu(r) \cos \nu\pi - r^\nu J_{-\nu}(r)\}/ \sin \nu\pi \tag{46}$$

when ν is not an integer. It follows that

$$(\nabla^2 + 1) r^\nu Y_\nu(r) = (2\nu + n - 2) r^{\nu-1} Y_{\nu-1}(r), \tag{47}$$

$$\sum_{j=1}^{n} x_j \partial_j r^\nu Y_\nu(r) = r^2 . r^{\nu-1} Y_{\nu-1}(r), \tag{48}$$

$$\sum_{j=1}^{n} x_j \partial_j r^\nu Y_\nu(r) - 2\nu r^\nu Y_\nu(r) = -r^{\nu+1} Y_{\nu+1}(r). \tag{49}$$

When ν is an integer the limit of (46) is used as a definition. There is no problein when ν is a positive integer and (47–49) continue to hold. Further consideration is necessary when ν is a negative integer because it may coincide with one of the exceptional values. The relevant definition is

$$\begin{aligned} r^{-k} Y_{-k}(r) = {} & \frac{(-1)^k}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+k)!2^{k+2p}} \left[2r^{2p} \log r - \{\psi(p) + \psi(p+k) + 2\log 2\} r^{2p}\right] \\ & - \frac{(-2)^k}{\pi} \sum_{p=0}^{k} \frac{(k-p-1)!}{p!2^{2p}} r^{2p-2k}. \end{aligned} \tag{50}$$

As a result

$$\begin{aligned} (\nabla^2 + 1) r^{-n/2-k} Y_{-n/2-k}(r) = {} & -2(k+1) r^{-n/2-k-1} Y_{-n/2-k-1}(r) \\ & + (-1)^k 2^{n/2-k-1} \pi^{n/2-1} \cos \tfrac{1}{2} n\pi \sum_{m=0}^{k+1} \frac{(n+4k-4m+2)(\nabla^2)^{k-m+1} \delta(\boldsymbol{x})}{m!(k+1-m)!(n/2+k-m)}. \end{aligned} \tag{51}$$

Furthermore

$$\begin{aligned} \sum_{j=1}^{n} x_j \partial_j r^{-n/2-k} Y_{-n/2-k}(r) = {} & r^2 . r^{-n/2-k-1} Y_{-n/2-k-1}(r) \\ & - (-1)^k 2^{n/2-k+1} \pi^{n/2-1} \cos \tfrac{1}{2} n\pi \sum_{m=0}^{k} \frac{(\nabla^2)^{k-m} \delta(\boldsymbol{x})}{m!(k-m)!}. \end{aligned} \tag{52}$$

Thus $r^\nu Y_\nu$ can serve as a basis for a fundamental solution when $r^\nu J_\nu$ is not available.

The corresponding Hankel functions are defined in the customary way, namely

$$\begin{aligned} r^\nu H_\nu^{(1)}(r) &= r^\nu J_\nu(r) + \mathrm{i} r^\nu Y_\nu(r), \\ r^\nu H_\nu^{(2)}(r) &= r^\nu J_\nu(r) - \mathrm{i} r^\nu Y_\nu(r). \end{aligned}$$

They can replace $r^\nu Y_\nu(r)$ in (47–49).

One result concerning Fourier transforms is

THEOREM 6.1. *When* β *is not an integer the Fourier transform of* $(r^2 - 1 + \mathrm{i}0)^\beta$ *is*

$$\frac{2^{n/2+\beta}\pi^{n/2+1}}{(-\beta-1)!} \mathrm{e}^{(\beta+n/2-1/2)\pi \mathrm{i}} \alpha^{-\beta-n/2} H^{(2)}_{-\beta-n/2}(\alpha).$$

For the transform of $(r^2 - 1 - \mathrm{i}0)^\beta$ *change the sign of* i *throughout.*

Proof. Assume firstly that $0 > \mathfrak{Re}(\beta) > -1$. Then $(r^2-1+\mathrm{i}0)^{\beta/2}$ is a conventional function and so is $K_{-\beta}\{\mu(r^2-1+\mathrm{i}0)^{1/2}\}$ where K_ν is the usual modified Bessel function. With $\mu > 0$, the function $(r^2 - 1 + \mathrm{i}0)^{\beta/2} K_{-\beta}\{\mu(r^2 - 1 + \mathrm{i}0)^{1/2}\}$ is absolutely integrable and its Fourier transform is

$$\begin{aligned} &\frac{(2\pi)^{n/2}}{\alpha^{n/2-1}} \int_0^\infty (r^2 - 1 + \mathrm{i}0)^{\beta/2} K_{-\beta}\{\mu(r^2 - 1 + \mathrm{i}0)^{1/2}\} r^{n/2} J_{n/2-1}(\alpha r)\mathrm{d}r \\ &= 2^{n/2-1}\pi^{n/2-1}\mathrm{e}^{(\beta+n/2-1/2)\pi\mathrm{i}}\mu^\beta(\mu^2+\alpha^2)^{-(\beta+n/2)/2} H^{(2)}_{-\beta-n/2}\left\{(\mu^2+\alpha^2)^{1/2}\right\} \end{aligned}$$

by virtue of a formula given by Watson [4, p. 416]. The transform in the theorem follows now provided that, for $0 < \mathfrak{Re}(\nu) < 1$,

$$\lim_{\mu\to 0}(\mu x)^\nu K_\nu(\mu x) = (\nu - 1)! 2^{\nu-1} \tag{53}$$

because only conventional multiplication occurs.

To verify (53) start with

$$x^\nu K_\nu(x) = (\nu - \tfrac{1}{2})! \frac{2^\nu}{\pi^{1/2}} \int_0^\infty \frac{\cos xt}{(t^2+1)^{\nu+1/2}} \mathrm{d}t \tag{54}$$

which is valid for $\mathfrak{Re}(\nu) > -1/2$ and $x > 0$. Also, if $\mathfrak{Re}(\nu) > 0$,

$$\int_0^\infty \frac{\mathrm{d}t}{(t^2+1)^{\nu+1/2}} = \frac{(\nu-1)!\pi^{1/2}}{(\nu-1/2)!2}$$

after the change of variable $t = \{u/(1-u)\}^{1/2}$. Hence

$$x^\nu K_\nu(x) - (\nu-1)!2^{\nu-1} = (\nu - \frac{1}{2})! \frac{2^\nu}{\pi^{1/2}} \int_0^\infty \frac{\cos xt - 1}{(t^2+1)^{\nu+1/2}} \mathrm{d}t.$$

Since $|\cos xt - 1| < xt$ it follows that

$$x^\nu K_\nu(x) - (\nu-1)!2^{\nu-1} = o(1) \tag{55}$$

as $x \to 0$ when $\mathfrak{Re}(\nu) > 1/2$. It is evident from (54) that $x^\nu K_\nu(x)$ is bounded when $\mathfrak{Re}(\nu) > 0$. Moreover $K_{-\nu}(x) = K_\nu(x)$. Hence, when $0 < \mathfrak{Re}(\nu) \le 1/2$,

$$2\nu x^\nu K_\nu(x) = x^{\nu+1}\{K_{\nu+1}(x) - K_{\nu-1}(x)\} = x^{\nu+1}\{K_{\nu+1}(x) - K_{1-\nu}(x)\}$$

shows that (55) still holds on applying (55) to the term involving $K_{\nu+1}$ and invoking the boundedness of $x^{1-\nu}K_{1-\nu}$. Thus, in fact, (55) is valid for $\Re\mathfrak{e}(\nu) > 0$ and (53) follows in a trivial fashion.

Accordingly, the theorem has been established for $0 > \Re\mathfrak{e}(\beta) > -1$. By (34) the Fourier transform of $(r^2 - 1 + \mathrm{i}0)^{\beta+1}$ is $-\tilde{\nabla}^2 - 1$ of the Fourier transform of $(r^2 - 1 + \mathrm{i}0)^{\beta}$. On account of (47) this turns out to be the same formula with β replaced by β + 1. Thus the transform is valid for $\Re\mathfrak{e}(\beta) > -1$, integers excepted. By virtue of (32) the Fourier transform of $(r^2 - 1 + \mathrm{i}0)^{\beta-1}$ is obtained from that of $(r^2 - 1 + \mathrm{i}0)^{\beta}$ by applying the operator

$$-\frac{1}{2\beta}\left\{\sum_{j=1}^{n}\alpha_j\tilde{\partial}_j + n + 2\beta\right\}.$$

Then (49) recovers the original with β replaced by β − 1. Hence the theorem has been demonstrated for $(r^2 - 1 + \mathrm{i}0)^{\beta}$ so long as β is not an integer.

For $(r^2 - 1 - \mathrm{i}0)^{\beta}$ it is necessary only to change the sign of i in the exponential and put $H^{(1)}$ for $H^{(2)}$. Thereafter the analysis follows the same route and the proof is concluded. ■

Fourier transforms of the Hankel funcfions can be deduced by Fourier inversion and from these the transforms of Bessel functions derived. They are given in

THEOREM 6.2. *When $\nu + n/2$ is not an integer the Fourier transform of $r^\nu J_\nu(r)$ is*

$$(\nu + \tfrac{1}{2}n - 1)!\mathrm{i}2^{n+\nu-1}\pi^{n/2-1}\left\{\mathrm{e}^{\nu\pi\mathrm{i}}(\alpha^2 - 1 + \mathrm{i}0)^{-\nu-n/2} - \mathrm{e}^{-\nu\pi\mathrm{i}}(\alpha^2 - 1 - \mathrm{i}0)^{-\nu-n/2}\right\}$$

and of $r^\nu Y_\nu(r)$ is

$$(\nu + \tfrac{1}{2}n - 1)!(-1)2^{n+\nu-1}\pi^{n/2-1}\left\{\mathrm{e}^{\nu\pi\mathrm{i}}(\alpha^2 - 1 + \mathrm{i}0)^{-\nu-n/2} + \mathrm{e}^{-\nu\pi\mathrm{i}}(\alpha^2 - 1 - \mathrm{i}0)^{-\nu-n/2}\right\}.$$

THEOREM 6.3. *With m a positive integer the Fourier transform of $r^{m-n/2}J_{m-n/2}(r)$ is*

$$(-1)^m 2^{m+n/2}\pi^{n/2-1}\left\{(m-1)!(\alpha^2 - 1)^{-m}\sin\tfrac{1}{2}n\pi + (-1)^{m-1}\pi\delta^{(m-1)}(\alpha^2 - 1)\cos\tfrac{1}{2}n\pi\right\}$$

and of $r^{m-n/2}Y_{m-n/2}(r)$ is

$$(-1)^{m-1}2^{m+n/2}\pi^{n/2-1}\left\{(m-1)!(\alpha^2 - 1)^{-m}\cos\tfrac{1}{2}n\pi + (-1)^{m-1}\pi\delta^{(m-1)}(\alpha^2 - 1)\sin\tfrac{1}{2}n\pi\right\}.$$

Proof. Since

$$\operatorname*{Lim}_{\mu\to 0} r^{\mu+m-n/2}J_{\mu+m-n/2}(r) = r^{m-n/2}J_{m-n/2}(r)$$

the formulae can be derived at once from Theorem 6.2, (36) and (35). ■

The remaining exceptional orders are dealt with by

THEOREM 6.4. *The Fourier transform of $r^{-n/2-k}J_{-n/2-k}(r)$ is*

$$2^{n/2-k}\frac{\pi^{n/2-1}}{k!}(\alpha^2 - 1)^k\left[\left\{\psi(k) + 2\log 2 - \log\left|(\alpha^2 - 1)\right|\right\}\sin\tfrac{1}{2}n\pi + \pi H(1-\alpha^2)\cos\tfrac{1}{2}n\pi\right]$$
$$+2^{n/2-k}\pi^{n/2-k}\sin\tfrac{1}{2}n\pi\sum_{m=0}^{k}\frac{(-1)^m\alpha^{2k-2m}}{m!(k-m)!}\psi(k + n/2 - m - 1)$$

and of $r^{-n/2-k}Y_{-n/2-k}(r)$ *is*

$$-2^{n/2-1}\frac{\pi^{n/2-1}}{k!}(\alpha^2-1)^k\left[\left\{\psi(k)+2\log 2-\log\left|(\alpha^2-1)\right|\right\}\cos\tfrac{1}{2}n\pi-\pi H(1-\alpha^2)\sin\tfrac{1}{2}n\pi\right]$$
$$-2^{n/2-k}\pi^{n/2-1}\cos\tfrac{1}{2}n\pi\sum_{m=0}^{k}\frac{(-1)^m\alpha^{2k-2m}}{m!(k-m)!}\psi(k+n/2-m-1).$$

Proof. Because

$$r^{-n/2-k}J_{-n/2-k}(r)=\operatorname*{Lim}_{\mu\to 0}\left\{r^{\mu-n/2-k}J_{\mu-n/2-k}(r)\right.$$
$$\left.-\sum_{m=0}^{k}\frac{2^{n/2-k-\mu}\pi^{n/2}(\nabla^2)^{k-m}\delta(\mathbf{x})(-1)^m}{m!(\mu-n/2-k+m)!(n/2+k-m-1)!(k-m)!\mu}\right\} \quad (56)$$

the transform of the left-hand side can be obtained by taking the limit of the transform on the right. The transform of the Bessel function on the rigid is available from Theorem 6.2 and the limit leads to the stated result.

For the other Bessel function the formula analogous to (56) is

$$r^{-n/2-k}Y_{-n/2-k}(r)=\operatorname*{Lim}_{\mu\to 0}\left\{r^{\mu-n/2-k}Y_{\mu-n/2-k}(r)\right.$$
$$\left.+(-1)^k2^{n/2-k-\mu}\pi^{n/2-1}\cos(\mu-\tfrac{1}{2}n)\pi\sum_{m=0}^{k}\frac{(n/2+k-m-\mu-1)!(\nabla^2)^{k-m}\delta(\boldsymbol{x})}{(n/2+k-m-1)!m!(k-m)!\mu}\right\}. \quad (57)$$

Again Theorem 6.2 supplies the transorm of the right-hand side and the theorem is proved. ■

Several other transforms can be inferred from the foregoing. For convenience they are listed in Appendix B without details of their derivation.

7. Continuity and asymptotic behaviour

In this section is studied the relation between the singular behaviour of a generalised function and the properties of its Fourier transform. The subject was investigated for generalised functions of a single variable by Lighhill [1] for the direct transform and by Lighthill [5] for the inverse transform. Here his ideas are extended to generalised functions of several variables.

THEOREM 7.1. *Let* $g(\mathbf{x})$ *and* $g_m(\mathbf{x})$ *be locally (absolutely) integrable and* $g_m(\mathbf{x})$ *be such that* $g(\mathbf{x})-g_m(\mathbf{x})$ *is absolutely integrable for* $r>R$*. Then, if* $G_m(\boldsymbol{\alpha})\to 0$ *as* $\alpha\to\infty$, $G(\boldsymbol{\alpha})\to 0$ *as* $\alpha\to\infty$.

Proof. The hypotheses ensure that $g(\mathbf{x})-g_m(\mathbf{x})$ is absolutely integrable. Therefore its transform $G(\boldsymbol{\alpha})-G_m(\boldsymbol{\alpha})\to 0$ as $\alpha\to\infty$ by the Riemann-Lebesgue lemma. Since $G_m(\boldsymbol{\alpha})\to 0$ as $\alpha\to\infty$ the theorem follows. ■

The theorem is useful because $g_m(\mathbf{x})$ can be chosen to be r^β or $r^\beta\log^m r$ with $\Re\mathfrak{e}(\beta)>-n$. They satisfy the conditions of the theorem on account of Theorem 4.1, Theorem 4.2, Theorem 4.4 and Theorem 4.6. This is still true if r^β is replaced by $\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}r^\beta$, with $\mathbf{a}$ a real constant vector, since the transform is merely translated by a finite amount. Evidently, $g(\mathbf{x})$

is not obliged to be absolutely integrable in $r > R$ for its Fourier transform to vanish at infinity. However, behaviour like r^{-n} and $r^{-n}\log^m r$ as $r \to \infty$ for $g(\mathbf{x})$ is excluded from the theorem because these generalised functions are not locally integrable and their transforms do not vanish at infinity (see Theorem 4.3 and Theorem 4.5). Nevertheless, they can be included by the next theorem.

THEOREM 7.2. *Let the only singularities of* $g_0(\mathbf{x})$ *be in* $r < R/2$ *and* $\partial_j g_0(\mathbf{x})$ *be absolutely integrable in* $r > R$ *for* $j = 1, \dots, n$. *Then* $g_0(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}H(r-R)$ *can be included in* $g_m(\mathbf{x})$ *in Theorem 7.1.*

Proof. Let $\eta(\mathbf{x})$ be an infinitely differentiable function which is unity for $r \geq R$ and zero for $r \leq R/2$. Then the function $g_0(\mathbf{x})\eta(\mathbf{x}) - g_0(\mathbf{x})H(r-R)$ vanishes for $r > R$ and $r \leq R/2$. It is finite elsewhere and so is absolutely integrable. Hence its Fourier transform tends to zero as a $\alpha \to \infty$. Therefore the desired result follows if the Fournier transform of $\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}g_0(\mathbf{x})\eta(\mathbf{x})$ tends to zero. Now

$$\partial_j\left\{\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}g_0(\mathbf{x})\eta(\mathbf{x})\right\} - \mathrm{i}a_j\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}g_0(\mathbf{x})\eta(\mathbf{x}) = \left\{\eta(\mathbf{x})\partial_j g_0(\mathbf{x}) + g_0(\mathbf{x})\partial_j\eta(\mathbf{x})\right\}\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}.$$

The given conditions ensure that the right-hand side is absolutely integrable so that its Fourier transform tends to zero as a $\alpha \to \infty$. Hence the left-hand side enjoys the same property. Accordingly, $\mathrm{i}(\alpha_j - a_j)$ times the Fourier transform of $\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}g_0(\mathbf{x})\eta(\mathbf{x})$ tends to zero. Whatever the direction of the radius vector to α at least one of $|\alpha_j|$ tends to inifinity as $\alpha \to \infty$. Therefore the Fourier transform of $\mathrm{e}^{\mathrm{i}\mathbf{a}\cdot\mathbf{x}}g_0(\mathbf{x})\eta(\mathbf{x})$ tends to zero as $\alpha \to \infty$ and the proof is complete. ■

There is an alternative version of Theorem 7.2 which is more symmetrical but is somewhat more restrictive in its conditions. One advantage is that it allows for a different exponential multiplier. It may be proved in the same way as Theorem 7.2 and is contained in

THEOREM 7.3. *Let the only singularities of* $g_0(\mathbf{x})$ *be in* $r < R/2$ *and* $(\nabla^2 + a^2)\{\mathrm{e}^{\mathrm{i}ar}g_0(\mathbf{x})\}$ *(a real) be absolutely integrable in* $r > R$. *Then* $\mathrm{e}^{\mathrm{i}ar}g_0(\mathbf{x})H(r-R)$ *can be included in* $g_m(\mathbf{x})$ *in Theorem 7.1.*

The generalised functions r^β and $r^\beta\log^m r$ with $\Re\mathfrak{e}(\beta) \geq -n$ are not the only possible candidates for g_m in Theorem 7.1. The generalised function $r^{-\nu}J_\nu(r)$ is locally integrable and, by the preceding section, its Fourier transform is identically zero in $\alpha > 1$ when $\Re\mathfrak{e}(\nu) > -n/2$ Therefore, $r^{-\nu}J_\nu(r)$ is acceptable for g_m when $\Re\mathfrak{e}(\nu) > -n/2$. Two other candidates when $\Re\mathfrak{e}(\nu) > -n/2$ are $r^\nu J_\nu(r)$ and $r^\nu Y_\nu(r)$. They are locally integrable and their Fourier transforms tend to zero as $\alpha \to \infty$ on account of Theorem 6.2 and Theorem 6.3. However, they are unacceptable when $\Re\mathfrak{e}(\nu) \leq -n/2$ because their Fourier transforms do not tend to zero as $\alpha \to \infty$ and, in addition, they are not locally integrable. Nor does the device of Theorem 7.2 help. Although it resolves the problem of local integrability, the oscillatory behaviour of J_ν and Y_ν at infinity precludes satisfaction of the condition on $\partial_j g_0$. Thus $r^\nu J_\nu(r)$ and $r^\nu Y_\nu(r)$ are available for Theorem 7.1 only when $\Re\mathfrak{e}(\nu) > -n/2$.

Note that because of the asymptotic behaviour of J_ν and Y_ν the generalised functions which behave like $\mathrm{e}^{\mathrm{i}ar}r^\beta$ (a real) at infinity are acceptable candidates provided that they comply with the other conditions imposed. Direct confirmation is forthcoming (for a fuller discussion see Appendix A). In the integral at the beginning of the proof of Theorem 4.2 replace μ by $\mu + \mathrm{i}a$. After the limit as $\mu \to 0$ is taken it is clear that the Fourier transform $\mathrm{e}^{\mathrm{i}ar}r^\beta$ tends to zero as $\alpha \to \infty$ when $\Re\mathfrak{e}(\beta) > -n$. That behaviour like $\mathrm{e}^{\mathrm{i}ar}r^{-n}$ at infinity can be permitted is a consequence of Theorem 7.3.

This direct confirmation means that it is possible to deduce something about the Fourier transform of $r^{-\nu}J_\nu(r)$ without knowledge of the earlier theorems. The asymptotic development of J_ν gives

$$r^{-\nu}J_\nu(r) = \frac{1}{(2\pi)^{1/2}r^{\nu+1/2}}\left\{e^{i(r-\nu\pi/2-\pi/4)} + e^{-i(r-\nu\pi/2-\pi/4)}\right\} + O\left(\frac{1}{r^{\nu+3/2}}\right)$$

as $r \to \infty$, Consequently $r^{-\nu}J_\nu(r)$ is absolutely integrable for $\mathfrak{Re}(\nu) > n-1/2$ and its Fourier transform tends to zero as $\alpha \to \infty$. On the other hand the order term is absolutely integrable for $\mathfrak{Re}(\nu) > n - 3/2$ and the first term above is of the type just considered. It is evident then that Theorem 7.1 makes the transform of $r^{-\nu}J_\nu(r)$ tend to zero for $\mathfrak{Re}(\nu) > n - 3/2$. The range of ν for which this is true can be extended by taking further terms in the asymptotic expansion of J_ν.

In order to deal with singularities at points other than the origin the following definition is introduced.

DEFINITION 7.1. *A generalised function is said to have a finite number of isolated singularities* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M$ *if it is equal to an infinitely differentiable function except at these points.*

Conventional functions with isolated singularities are not uncommon. The relation between the behaviour of the Fourier transform and the singularities is a matter of some interest. Of course it makes sense to treat them as generalised functions since their conventional transforms may not exist.

DEFINITION 7.2. *A generalised function* $g(\mathbf{x})$ *such that* $g(\mathbf{x})-g_m(\mathbf{x})$ *is absolutely integrable over* $r > R$ *is said to be well-behaved at infinity if* $G_m(\boldsymbol{\alpha}) \to 0$ *as* $\alpha \to \infty$.

We can state now

THEOREM 7.4. *Let* $g(\mathbf{x})$ *possess a finite number of isolated singularities at* $\mathbf{x}_1, \ldots, \mathbf{x}_M$. *For* $m = 1, \ldots, M$ *let* $h_m(\mathbf{x})$ *be infinitely differentiable except for a singularity at* $\mathbf{x}_m$. *Suppose that* $\partial_j^N\{g(\mathbf{x}) - h_m(\mathbf{x})\}$ *is absolutely integrable over some sphere which encloses* $\mathbf{x}_m$ *for* $m = 1, \ldots, M$. *Then, if* $\partial_j^N g(\mathbf{x})$ *and* $\partial_j^N h_m(\mathbf{x})$ *are well-behaved at infinity,*

$$G(\boldsymbol{\alpha}) = \sum_{m=1}^{M} H_m(\boldsymbol{\alpha}) + o\left(|\alpha_j|^{-N}\right)$$

as $\alpha_j \to \infty$.

Proof. Since $g(\mathbf{x})$ has only a finite nunber of singularities, $\partial_j^N g(\mathbf{x})$ is absolutely integrable over any finite domain which excludes the singularities. Also $\partial_j^N h_m(\mathbf{x})$ is absolutely integrable over any finite domain which excludes $\mathbf{x}_m$. On the other hand, $\partial_j^N\{g(\mathbf{x}) - h_m(\mathbf{x})\}$ is absolutely integrable over some domain containing $\mathbf{x}_m$. Hence $\partial_j^N\{g(\mathbf{x}) - h_m(\mathbf{x})\}$ is absolutely integable over any finite domain which excludes $\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_M$. Therefore $\partial_j^N\{g(\mathbf{x}) - \sum_{m=1}^M h_m(\mathbf{x})\}$ is locally integrable. It is well-behaved at infinity by hypothesis. Consequently, Theorem 7.1 ensures that its Fourier transform tends to zero as $\alpha \to \infty$ *i.e.*

$$(i\alpha_j)^N\left\{G(\boldsymbol{\alpha}) - \sum_{m=1}^{M} H_m(\boldsymbol{\alpha})\right\} = o(1).$$

The theorem follows by choosing the direction of $\boldsymbol{\alpha}$ to be the α_j-axis. ∎

In essence the theorem states that the singularities of $g(\mathbf{x})$ determine the asymptotic behaviour of its Fourier transform. Naturally, it may not be possible to have the same N for all j. The magnitude of the error may vary as the point of observation moves about the $\boldsymbol{\alpha}$-space. Uniform behaviour can be obtained by requiring a little more of $g(\mathbf{x})$. An obvious modification of the proof of Theorem 7.4 gives

THEOREM 7.5. *If the conditions of Theorem 7.4 are satisfied except that ∂_j^N is replaced by $(\nabla^2)^N$ then*

$$G(\boldsymbol{\alpha}) = \sum_{m=1}^{M} H_m(\boldsymbol{\alpha}) + o\left(\alpha^{-2N}\right).$$

The function $r^\nu J_\nu(r)$ with $\Re\mathfrak{e}(\nu) > -n/2$ has as single singularity at the origin. If $h_1(\mathbf{x}) = r^{2\nu}/\nu!2^\nu$ the conditions of Theorem 7.4 are met for any j provided that N is any positive integer such that $N < 2\nu + 2 + n$. Therefore, by Theorem 4.2, the Fourier transform of $r^\nu J_\nu(r)$ is

$$(\nu + n/2 - 1)!(-1)2^{\nu+n}\pi^{n/2-1}\alpha^{-n-2\nu}\sin\nu\pi + o\left(\alpha^{-N}\right)$$

as $\alpha \to \infty$. This is consistent with Theorem 6.2. If extra terms in the expansion of $r^\nu J_\nu(r)$ near the origin were included in $h_1(\mathbf{x})$ the value of N could be increased and further terms in the asymptotic development of Theorem 6.2 in powers of α^{-2} obtained.

Of course, the singularity could be moved from the origin by considering $|\mathbf{x} - \mathbf{x}_1|^\nu J_\nu(|\mathbf{x} - \mathbf{x}_1|)$ say. The only effect is to multiply $H_1(\boldsymbol{\alpha})$ by $e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}_1}$ without altering the error estimate.

As explained already, the singularities of $g(\mathbf{x})$ dictate the behaviour of the Fourier transform. A Fourier inversion suggests that the behaviour of $g(\mathbf{x})$ at infinity is responsible for the singularities of its transform. Whether or not this suggestion can be justified is the next topic to be discussed.

THEOREM 7.6. *If the generalised function $g(\mathbf{x})$ is absolutely integrable then $G(\boldsymbol{\alpha})$ is locally contimous in the sense that, given $\delta > 0$, there is an $\varepsilon > 0$ such that $|G(\boldsymbol{\alpha} + \boldsymbol{\beta}) - G(\boldsymbol{\alpha})| < \delta$ for $|\boldsymbol{\beta}| < \varepsilon$.*

Proof.

$$G(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha} + \boldsymbol{\beta}) = 2i \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i(\boldsymbol{\alpha}+\boldsymbol{\beta}/2)\cdot\mathbf{x}} \sin \tfrac{1}{2}\boldsymbol{\beta}\cdot\mathbf{x}\,d\mathbf{x}.$$

Hence

$$\begin{aligned} |G(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha} + \boldsymbol{\beta})| &\leq 2\int_{-\infty}^{\infty} |g(\mathbf{x})| \left|\sin \tfrac{1}{2}\boldsymbol{\beta}\cdot\mathbf{x}\right| d\mathbf{x} \\ &\leq |\boldsymbol{\beta}| \int_{r\leq R} |g(\mathbf{x})|\,|\mathbf{x}|\,d\mathbf{x} + 2\int_{r>R} |g(\mathbf{x})|\,d\mathbf{x}. \end{aligned}$$

The second integral on the right can be made less than $\delta/4$ by choosing R large enough since g is absolutely integrable. With R fixed, the first term can be made less than $\delta/2$ by selecting $|\boldsymbol{\beta}|$ small enough. The proof is concluded. ∎

The analogue of Theorem 7.1 is

THEOREM 7.7. *Let $g(\mathbf{x})$ and $g_m(\mathbf{x})$ be absolutely integrable over $r > R$ and let $g(\mathbf{x}) - g_m(\mathbf{x})$ be locally integrable. Then, if $G_m(\boldsymbol{\alpha})$ is locally continuous, so is $G(\boldsymbol{\alpha})$.*

Proof. The given conditions make $g(\mathbf{x}) - g_m(\mathbf{x})$ absolutely integrable. From Theorem 7.6 $G(\boldsymbol{\alpha}) - G_m(\boldsymbol{\alpha})$ is locally continuous.
The theorem follows from the local continuity of $G_m(\boldsymbol{\alpha})$. ■

Both r^β and $r^\beta \log^m r$ are suitable choices for $g(\mathbf{x})$ when $\mathfrak{Re}(\beta) < -n$. But r^{-n} is not. It is not absolutely integrable over $r > R$ and its Fourier transform is not locally continuous as can be seen from Theorem 4.3. Nevertheless it can be accommodated by means of

THEOREM 7.8. *Let $x_j g_0(\mathbf{x})$ be locally integrable for $j = 1, \dots, n$. Then $g_0(\mathbf{x})H(R - r)$ can be included in $g_m(\mathbf{x})$ in Theorem 7.7.*

Proof. Since $x_j g_0(\mathbf{x})H(R - r)$ is locally integrable and vanishes for $r > R$ it is absolutely integrable. Therefore its transform is locally continuous. Consequently, $\tilde{\partial}_j \tilde{G}_0(\boldsymbol{\alpha})$, where $\tilde{G}_0(\boldsymbol{\alpha})$ is the transform of $g_0(\mathbf{x})H(R - r)$, is locally continuous for every j. The continuity of grad $\tilde{G}_0$ implies that of $\tilde{G}_0$ and the proof is concluded. ■

To relate the singularities of transforms we introduce

DEFINITION 7.3. *A generalised function $g(\mathbf{x})$ such that $g(\mathbf{x}) - g_m(\mathbf{x})$ is locally integrable is said to be locally well-behaved if $G_m(\boldsymbol{\alpha})$ is locally continuous.*

Then we have

THEOREM 7.9. *Let $g(\mathbf{x})$ and $h_m(\mathbf{x})$ be locally well-behaved. If $x_j^N\{g(\mathbf{x}) - h_m(\mathbf{x})\}$ (N a non-negative integer) is absolutely integrable over $r > R$ then $\tilde{\partial}_j^N\{G(\boldsymbol{\alpha}) - H_m(\boldsymbol{\alpha})\}$ is locally continuous.*

Proof. The statement is an immediate consequence of the conditions imposed and Theorem 7.7. ■

In effect Theorem 7.9 demonstrates that $G(\boldsymbol{\alpha})$ and $H_m(\boldsymbol{\alpha})$ exhibit the same kind of singular behaviour. Another interpretation is that the behaviour of $g(\mathbf{x})$ at infinity determines the singularities of $G(\boldsymbol{\alpha})$. The theorem may be regarded as an inverse transform of Theorem 7.4.

An illustration of Theorem 7.9 is provided by taking $g(\mathbf{x}) = r^\nu J_\nu(r)$. This generalised function is locally well-behaved because any terms which are not locally integrable may be removed by $g_m(\mathbf{x})$; the local continuity of $G_m(\boldsymbol{\alpha})$ follows from Theorem 7.8, Theorem 4.2 and Theorem 4.3.

For larger r

$$g(\mathbf{x}) = \frac{r^{\nu-1/2}}{(2\pi)^{1/2}}\left\{e^{ir-(\nu+1/2)\pi i/2} + e^{-ir-(\nu+1/2)\pi i/2}\right\} + O\left(r^{\nu-3/2}\right). \tag{58}$$

The generalised functions in (58) are locally well-behaved by the same argument as was used for $r^\nu J_\nu(r)$. The order term is absolutely integrable over $r > R$ so long as $\nu < 3/2 - n$. Hence, by Theorem 7.9 with $N = 0$, the transform of $r^\nu J_\nu(r)$ has the same singularities as those of the transform of the generalised functions on the right-hand side of (58) when $\nu < 3/2 - n$.

The Fourier transforms of $e^{\pm ir} r^\beta$ are derived in Appendix A. By the remark at the end of the appendix their dominant behaviour is available from (A.8) if $\beta \neq -n/2 - 1/2 - k$ and (A.10) if $\beta = -n/2 - 1/2 - k$. Hence (A.8) may be quoted when $\nu \neq -n/2 - k$ and it is found that the singular behaviour coincides precisely with that furnished by Theorem 6.2 and Theorem 6.3. On the other hand, when $\nu = -n/2 - k$, (A.10) supplies singular terms in agreement with those in Theorem 6.4.

Although the illustration has verified Theorem 7.9 only for $\nu < 3/2 - n$ the result can be extended to higher values of ν by including more terms in the asymptotic development of $J_\nu(r)$. Thereby the order term is reduced so that ν may be increased while retaining integrability at infinity. According to Appendix A the extra terms have Fourier transforms which are less singular than that provided by (58). Consequently, the dominant behaviour is unaffected by the increase in ν and the restriction on its size can be dropped.

In principle, more terms in the expansion in the neighbourhood of a singularity can be obtained from the asymptotic development of $J_\nu(r)$. It would be necessary to calculate further terms in the transforms of $\mathrm{e}^{\pm ibr}r^\beta$ beyond those in the appendix where only the leading terms in the expansion have been set out.

Appendix A

In this appendix the generalised function $\mathrm{e}^{-cr}r^\beta$ is considered. The constant c is allowed to be complex subject to $\Re\mathfrak{e}(c) \geq 0$ so that $|\mathrm{ph}(c)| \leq \pi/2$. Initially it will be assumed that $\Re\mathfrak{e}(c) > 0$. Later the case when $\Re\mathfrak{e}(c) = 0$ will be discussed but, since $c = 0$ has been handled already in Section 2, the condition $c \neq 0$ is imposed throughout. The generalised function may be defined in an obvious way by

$$\mathrm{e}^{-cr}r^\beta = \sum_{m=0}^{\infty} \frac{(-c)^m}{m!} r^{\beta+m}. \tag{A.1}$$

its properties can then be deduced from those of r^β in Section 2. Thus, if $\beta \neq 2 - n - k$,

$$(\nabla^2 - c^2)\mathrm{e}^{-cr}r^\beta = \beta(\beta + n - 2)\mathrm{e}^{-cr}r^{\beta-2} - c(n + 2\beta - 1)\mathrm{e}^{-cr}r^{\beta-1} \tag{A.2}$$

from Definition 2.1 and, if $\beta \neq -n - k$,

$$\partial_j \mathrm{e}^{-cr}r^\beta = \beta x_j \mathrm{e}^{-cr}r^{\beta-2} - c x_j \mathrm{e}^{-cr}r^{\beta-1} \tag{A.3}$$

from (5). On the other hand, (13) supplies

$$\begin{aligned}(\nabla^2 - c^2)\mathrm{e}^{-cr}r^{2-n-2k} &= 2k(2k + n - 2)\mathrm{e}^{-cr}r^{-n-2k} - c(3 - n - 4k)\mathrm{e}^{-cr}r^{1-n-2k} \\ &\quad - \sum_{m=0}^{k} \frac{c^{2m}}{(2m)!} \frac{(n + 4k - 4m - 2)\pi^{n/2}(\nabla^2)^{k-m}\delta(\mathbf{x})}{(n/2 + k - m - 1)!(k - m)!2^{2k-2m-1}}\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}(\nabla^2 - c^2)\mathrm{e}^{-cr}r^{1-n-2k} &= (2k + 1)(2k + n - 1)\mathrm{e}^{-cr}r^{-1-n-2k} - c(1 - n - 4k)\mathrm{e}^{-cr}r^{-n-2k} \\ &\quad + \sum_{m=0}^{k} \frac{c^{2m+1}}{(2m + 1)!} \frac{(n + 4k - 4m - 2)\pi^{n/2}(\nabla^2)^{k-m}\delta(\mathbf{x})}{(n/2 + k - m - 1)!(k - m)!2^{2k-2m-1}}.\end{aligned} \tag{A.5}$$

Also

$$\begin{aligned}\partial_j \mathrm{e}^{-cr}r^{-n-2k} &= (-n + 2k)x_j \mathrm{e}^{-cr}r^{-n-2k-2} - c x_j \mathrm{e}^{-cr}r^{-n-2k-1} \\ &\quad + \sum_{p=0}^{k} \frac{c^{2p}}{(2p)!} \frac{x_j(\nabla^2)^{k-p+1}\delta(\mathbf{x})}{(n/2 + k - p)!(k - p + 1)!2^{2k-2p+1}}\end{aligned} \tag{A.6}$$

and

$$\partial_j \mathrm{e}^{-cr} r^{-n-2k-1} = -(n+2k+1)x_j \mathrm{e}^{-cr} r^{-n-2k-3} - cx_j \mathrm{e}^{-cr} r^{-n-2k-2} - \sum_{p=0}^{k} \frac{c^{2p+1}}{(2p+1)!} \frac{x_j (\nabla^2)^{k-p+1} \delta(\mathbf{x})}{(n/2+k-p)!(k-p+1)!2^{2k-2p+1}}. \tag{A.7}$$

For the Fourier transform we have

THEOREM A.1. *If* $\beta \neq -n-k$ *the Fourier transform of* $\mathrm{e}^{-cr} r^{\beta}$ *is*

$$\frac{(\beta+n-1)!}{(n/2-1)!} 2\pi^{n/2} c^{-\beta-n} F\left(\tfrac{1}{2}\beta + \tfrac{1}{2}n, \tfrac{1}{2}\beta + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}n, -\frac{\alpha^2}{c^2}\right).$$

Proof. When $\Re\mathfrak{e}(\beta) > -n$ the transform has been calculated already in the proof of Theorem 4.2. It takes the form quoted after a standard transformation of the hypergeometric function. The formula can be extended to lower values of β by taking advantage of (A.2). This enables the transform of $\mathrm{e}^{-cr} r^{\beta-2}$ to be calculated from higher values of β. By drawing benefit from the relation

$$\begin{aligned} &c^2(n+2\beta-1)F\left(\tfrac{1}{2}\beta + \tfrac{1}{2}n, \tfrac{1}{2}\beta + \tfrac{1}{2}n - \tfrac{1}{2}, \tfrac{1}{2}n, -\tfrac{\alpha^2}{c^2}\right) \\ &- (c^2+\alpha^2)(\beta+n-1)F\left(\tfrac{1}{2}\beta + \tfrac{1}{2}n, \tfrac{1}{2}\beta + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}n, -\tfrac{\alpha^2}{c^2}\right) \\ &= c^2\beta F\left(\tfrac{1}{2}\beta + \tfrac{1}{2}n - 1, \tfrac{1}{2}\beta + \tfrac{1}{2}n - \tfrac{1}{2}, \tfrac{1}{2}n, -\tfrac{\alpha^2}{c^2}\right) \end{aligned}$$

it can be confirmed that the same formula is obtained with β replaced by β − 2. The proof is concluded. ■

The transform when β takes one of the values excluded by Theorem A.1 is slightly more complicated because there are two cases to consider

THEOREM A.2. *The Fourier transform of* $\mathrm{e}^{-cr} r^{-n-2k}$ *is*

$$\pi^{n/2} c^{2k} \sum_{m=0}^{k} \frac{\{\psi(2k-2m) - \log c\}\, z^m}{(2k-2m)!m!(n/2+m-1)!2^{2m-1}} + 2\pi^{n/2} c^{2k} \sum_{m=k+1}^{\infty} \frac{(2m-2k-1)! z^m}{m!(n/2+m-1)!2^{2m}}$$

and of $\mathrm{e}^{-cr} r^{-n-2k-1}$ *is*

$$-\pi^{n/2} c^{2k+1} \sum_{m=0}^{k} \frac{\{\psi(2k+1-2m) - \log c\}\, z^m}{(2k+1-2m)!m!(n/2+m-1)!2^{2m-1}} + 2\pi^{n/2} c^{2k+1} \sum_{m=k+1}^{\infty} \frac{(2m-2k-2)! z^m}{m!(n/2+m-1)!2^{2m}}$$

where $z = -\alpha^2/c^2$.

Proof. Since

$$\mathrm{e}^{-cr} r^{-n-2k} = \operatorname*{Lim}_{\mu\to 0} \left\{ \mathrm{e}^{-cr} r^{\mu-n-2k} - \sum_{p=0}^{k} \frac{c^{2p}}{(2p)!} \frac{\pi^{n/2} (\nabla^2)^{k-p} \delta(\mathbf{x})}{(n/2+k-p-1)!(k-p)!2^{2k-2p-1}\mu} \right\}$$

the Fourier transform of $e^{-cr}r^{-n-2k}$ is

$$\operatorname*{Lim}_{\mu\to 0}\left\{\frac{(\mu-2k-1)!}{(n/2-1)!}2\pi^{n/2}c^{2k-\mu}F\left(\tfrac{1}{2}\mu-k,\tfrac{1}{2}\mu-k+\tfrac{1}{2},\tfrac{1}{2}n,z\right)\right.$$
$$\left.-\sum_{p=0}^{k}\frac{c^{2p}\pi^{n/2}(-\alpha^2)^{k-p}}{(2p)!(n/2+k-p-1)!(k-p)!2^{2k-2p-1}\mu}\right\}$$

by Theorem A.1. When $|z|<1$ the hypergeometric function can be expanded in the usual series. The first k terms contain $1/\mu$ and, if μ is made zero in the remaining factor, the last series in {} is cancelled. Therefore the limit can be calculated as the derivative with respect to μ in the first k terms plus the sum of the remaining terms (with $\mu=0$) in the hypergeometric series. The result is the formula quoted in the theorem. Although demonstrated for $|z|<1$ it turns out (as will be verified shortly) that the infinite series is a regular function of z in the z-plane cut along the positive real axis from 1 to infinity. Accordingly, the restriction to $|z|<1$ can be dropped.

A parallel procedure gives the transform of $e^{-cr}r^{-n-2k-1}$ and the proof is finished. ■

Neither the transform of Theorem A.1 nor that of Theorem A.2 possesses a singularity unless α^2 approaches $-c^2$. This can never occur when $\Re\mathfrak{e}(c)>0$ as has been assumed hitherto. Therefore, what happens when $\Re\mathfrak{e}(c)=0$ will be considered now.

As $c\to ib$ (b positive) the point of observation may approach the branch line of the transforms obtained already. In order to stay on the principal branch it is necessary that $z(=-\alpha^2/c^2)$ be such that $0<\mathrm{ph}(z-1)<2\pi$ when $\Re\mathfrak{e}(c)\neq 0$. Hence, as $c\to ib$,

$$z-1\to\frac{\alpha^2}{b^2}-1+i0$$

and $1-z\to e^{-i\pi}(\alpha^2/b^2-1+i0)$ in any analytic continuation. Thus, we have

THEOREM A.3. *Under the conditions of Theorems A.1 and A.2 the Fourier transform of* $e^{-ibr}r^{\beta}$ *(*$b>0$*) is obtained by replacing* c *by* ib*,* $z-1$ *by* α^2/b^2-1+i0 *and* $1-z$ *by* $e^{-i\pi}(\alpha^2/b^2-1+i0)$. *For the transform of* $e^{ibr}r^{\beta}$ *change the sign of* i *throughout the replacements..*

Analytic continuation of the hypergeometric function when $\beta\neq -n-k$ shows that the singular part of the Fourier transform of $e^{-ibr}r^{\beta}$ is

$$(\beta+\tfrac{1}{2}n-\tfrac{1}{2})!2^{\beta+n}\pi^{(n-1)/2}(ib)^{-\beta-n}e^{(n/2+\beta+1/2)\pi i}\left(\tfrac{\alpha^2}{b^2}-1+i0\right)^{-n/2-\beta-1/2}$$
$$\times F\left(-\tfrac{1}{2}\beta,-\tfrac{1}{2}\beta-\tfrac{1}{2},\tfrac{1}{2}-\beta-\tfrac{1}{2}n,e^{-\pi i}(\alpha^2/b^2-1+i0)\right)$$

so long as $\beta+n/2+1/2$ is not an integer. Consequently, the dominant singularity of the transform of $e^{\pm ibr}r^{\beta}$ is

$$(\beta+\tfrac{1}{2}n-\tfrac{1}{2})!2^{\beta+n}\pi^{(n-1)/2}(\mp ib)^{-\beta-n}e^{\mp(n/2+\beta+1/2)\pi i}\left(\frac{\alpha^2}{b^2}-1\mp i0\right)^{-n/2-\beta-1/2} \tag{A.8}$$

when $\beta\neq -n-k$ and $\beta+n/2+1/2$ is not an integer. When $\beta+n/2+1/2$ is an integer the analytic continuation of the hypergeometric function takes a different form. If $\beta+n/2+1/2=m$ where m is a positive integer the dominant behaviour at the singularity of the transform of $e^{\pm ibr}r^{-n/2-1/2+m}$ is

$$(m-1)!2^{n/2+m-1/2}(-1)^m\pi^{(n-1)/2}(\mp ib)^{-m-n/2-1/2}(\alpha^2/b^2-1\mp i0)^{-m}. \tag{A.9}$$

It will be noticed that (A.9) is, in fact, the same as would be obtained from (A.8) by substitution of $\beta = -n/2 - 1/2 + m$. Of course that observation is limited to the dominant behaviour near the singularity. Inclusion of the terms of lower order near the singularity would bring logarithms into (A.9) but not (A.8).

When $\beta = -n/2 - 1/2 - k$ the restriction $k < n/2 - 1/2$ is imposed when n is odd to maintain β within the scope of Theorem A.1. With that understanding the analytic continuation of the hypergeometric function gives the dominant singular behaviour of the transform of $e^{\pm ibr} r^{-n/2-1/2-k}$ as

$$-2^{n/2-k-1/2}\frac{\pi^{(n-1)/2}}{k!}(\mp ib)^{k-n/2+1/2}\left(\frac{\alpha^2}{b^2}-1\right)^k\left\{\log\left|\frac{\alpha^2}{b^2}-1\right| \pm \pi i H(\alpha^2-b^2)\right\}. \quad \text{(A.10)}$$

When $\beta = -n - 2k$ the singularity (if any) of the transform is given by the infinite series in Theorem A.2. Since the dominant behaviour is dictated by the terms at infinity it can be identified with $\pi^{(n-1)/2}(\frac{1}{2}c)^{2k}\Phi(z, \frac{1}{2}n + \frac{1}{2} + 2k, 1)$ where

$$\Phi(z, s, 1) = \sum_{m=0}^{\infty}(m+1)^{-s}z^m.$$

The function $\Phi(z, s, 1)$ is regular in the z-plane cut along the positive real axis from 1 to ∞ when $\Re\mathfrak{e}(s) > 0$, which verifies an earlier statement. When s is not a positive integer the dominant singularity when z is near 1 is provided by

$$\Phi(z, s, 1) \simeq -\frac{(-s)!}{z}(e^{-\pi i}\log z)^{s-1}$$

and, when s is the positive integer m, by

$$\Phi(z, m, 1) \simeq -\frac{(\log z)^{m-1}}{(m-1)!z}\log(e^{-\pi i}\log z).$$

Hence, when n is even the dominant singular behaviour of the Fourier transform of $e^{\pm ibr}$ r^{-n-2k} is

$$(-\frac{1}{2}n - \frac{1}{2} - 2k)!\pi^{(n-1)/2}(-1)^k e^{\pm(n-1)\pi i/2}(\alpha^2 - b^2 \mp i0)^{n/2-1/2+2k}/2^{2k}b^{n-1+2k}. \quad \text{(A.11)}$$

When n is odd the corresponding result is

$$-\frac{(\alpha^2-b^2)^{n/2-1/2+2k}(-1)^k}{(\frac{1}{2}n-\frac{1}{2}+2k)!2^{2k}b^{n-1+2k}}\pi^{(n-1)/2}\left\{\log\left|\frac{\alpha^2}{b^2}-1\right| \pm \pi i H(\alpha^2-b^2)\right\}. \quad \text{(A.12)}$$

Likewise the dominant singular behaviour of the Fourier transform of $e^{\pm ibr}$ $r^{-n-2k-1}$ is

$$(-\tfrac{1}{2}n - \tfrac{1}{2} - 2k - 1)!\pi^{(n-1)/2}(\mp i)(-1)^{k+1}e^{\pm(n-1)\pi i/2} \times(\alpha^2 - b^2 \mp i0)^{n/2+1/2+2k}/2^{2k+1}b^{n+2k} \quad \text{(A.13)}$$

when n is even and

$$-\frac{(\alpha^2-b^2)^{n/2+1/2+2k}(-1)^k}{(\frac{1}{2}n+\frac{1}{2}+2k)!2^{2k+1}b^{n+2k}}(\mp i)\pi^{(n-1)/2}\left\{\log\left|\frac{\alpha^2}{b^2}-1\right| \pm \pi i H(\alpha^2-b^2)\right\} \quad \text{(A.14)}$$

when n is odd.

Observe that (A.11) and (A.13) are the same as would be given by (A.8) with $\beta = -n - 2k$ and $\beta = -n-2k-1$, respectively. Also (A.12) and (A.14) fit in with (A.10) if the restriction on k when n is odd is lifted. It may be concluded, therefore, that, *as far as the dominant singular behaviour of the transform of* $e^{\pm ibr} r^{\beta}$ *is concerned the formula* (A.8) *may be employed unless* $\beta = -n/2 - 1/2 - k$ *when* (A.10) *should be used.*

Appendix B

The following is a list of some Fourier transforms.

Table 1. Some Fourier transforms

g(**x**)	G(**α**)
$(r^2-1)_+^{\beta}$ β not an integer	$\beta!(-1)(2\pi)^{n/2}2^{\beta}\{\alpha^{-\beta-n/2}J_{-\beta-n/2}(\alpha)\cos\frac{1}{2}n\pi + \alpha^{-\beta-n/2}Y_{-\beta-n/2}(\alpha)\sin\frac{1}{2}n\pi\}$
$(r^2-1)_-^{\beta}$ β not an integer	$\beta!(2\pi)^{n/2}2^{\beta}\alpha^{-\beta-n/2}J_{\beta+n/2}(\alpha)$
$(r^2-1)_+^{k}$	$k!(-1)^{k+1}(2\pi)^{n/2}2^{k}\left\{\alpha^{-k-n/2}J_{k+n/2}(\alpha) - 2^{n/2-k}\pi^{n/2}\sum_{m=0}^{k}\frac{(\nabla^2)^{k-m}\delta(\boldsymbol{\alpha})}{m!(k-m)!}\right\}$
$(r^2-1)_-^{k}$	$k!(2\pi)^{n/2}2^{\beta k}\alpha^{-k-n/2}J_{k+n/2}(\alpha)$
$\delta^{(k)}(r^2-1)$	$(-1)^k(2\pi)^{n/2}\alpha^{k+1-n/2}J_{n/2-k-1}(\alpha)/2^{k+1}$
$(r^2-1)_+^{-m}$	$\frac{(2\pi)^{n/2}}{(m-1)!2^m}\left[\left\{\psi(m-1)-\log\frac{1}{2}\alpha\right\}\alpha^{m-n/2}J_{n/2-m}(\alpha) + (-1)^m\left\{\alpha^{m-n/2}\frac{\partial}{\partial\mu}J_{m-\mu-n/2}(\alpha)\cos\frac{1}{2}n\pi + \alpha^{m-n/2}\frac{\partial}{\partial\mu}Y_{m-\mu-n/2}(\alpha)\sin\frac{1}{2}n\pi\right\}_{\mu=0}\right]$
$(r^2-1)_-^{-m}$	$\frac{(2\pi)^{n/2}(-1)^{m-1}}{(m-1)!2^m}\left[\left\{\psi(m-1)-\log\frac{1}{2}\alpha\right\}\alpha^{m-n/2}J_{n/2-m}(\alpha) + \alpha^{m-n/2}\left\{\frac{\partial}{\partial\mu}J_{\mu-m+n/2}(\alpha)\right\}_{\mu=0}\right]$
$(r^2-1+i0)^{\beta}$ β not an integer	$2^{n/2+\beta}\pi^{n/2+1}e^{(\beta+n/2-1/2)\pi i}\alpha^{-\beta-n/2}H^{(2)}_{-\beta-n/2}(\alpha)/(-\beta-1)!$
$(r^2-1-i0)^{\beta}$ β not an integer	$2^{n/2+\beta}\pi^{n/2+1}e^{-(\beta+n/2-1/2)\pi i}\alpha^{-\beta-n/2}H^{(1)}_{-\beta-n/2}(\alpha)/(-\beta-1)!$

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